

network *inference*

introduction to *network analysis* (*ina*)

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# inference *overview*

- *inferring missing/spurious/hidden nodes/links*
  - due to *sampling, errors, noise* or *other* [GSP09, MBN15]
  - from *network structure, dynamics* or *other* [GRLK12]
- popular *predicting future links* that are likely to occur
  - recommendation of *friendship ties* on *Facebook* [BL11]
  - prediction of *product ratings* on *Amazon* [GLGMSP16]
  - prediction for costly *protein interaction* networks etc.



real, observed & reconstructed air transportation network [GSP09]

link *prediction*

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## prediction *overview*

which *links* are most *likely to occur*?

- link prediction by *local structure/dynamics*
  - *structural equivalence* [LW71] and *topological overlap* [RSM<sup>+</sup>02]
  - *node similarity* [LHN06] and *local dynamics* indices [ZLZ09]
- link prediction by *global structure/dynamics*
  - *regular equivalence* [WR83] and *link analysis* algorithms [JW02]
  - *community detection* [GN02] and *blockmodeling* [DBF05, Pei15]
- link prediction by *maximum likelihood* methods
  - *hierarchical* [CMN08] and *stochastic block* models [GSP09]
- link prediction by *probabilistic inference* methods
  - *probabilistic relational* models [FGKP99, SPH06]

## prediction *equivalence*

*links* predicted by *structural equivalence*

- *common neighbors index* [LW71] for  $i$  and  $j$  is

$$s_{ij} = \sum_x A_{ix}A_{xj} = |\Gamma_i \cap \Gamma_j|$$

- *Jaccard neighbors index* [Jac01] for  $i$  and  $j$  is

$$s_{ij} = \frac{\sum_x A_{ix}A_{xj}}{\sum_x A_{ix} + \sum_x A_{xj} - \sum_x A_{ix}A_{xj}} = \frac{|\Gamma_i \cap \Gamma_j|}{|\Gamma_i \cup \Gamma_j|}$$

- *Salton cosine similarity* [SM83] for  $i$  and  $j$  is

$$s_{ij} = \cos \theta_{ij} = \frac{\sum_x A_{ix}A_{xj}}{\sqrt{\sum_x A_{ix}^2} \sqrt{\sum_x A_{jx}^2}} = \frac{|\Gamma_i \cap \Gamma_j|}{\sqrt{k_i k_j}}$$

- *Leicht similarity index* [LHN06] for  $i$  and  $j$  is

$$s_{ij} = \frac{n \sum_x A_{ix}A_{xj}}{\sum_x A_{ix} \sum_x A_{jx}} = \frac{|\Gamma_i \cap \Gamma_j|}{k_i k_j / n} \approx \frac{|\Gamma_i \cap \Gamma_j|}{k_i k_j}$$

## prediction *overlap*

*links* predicted by *topological overlap*

- *Sørensen neighbors index* [Sør48] for  $i$  and  $j$  is

$$s_{ij} = \frac{\sum_x A_{ix}A_{xj}}{\frac{1}{2}(\sum_x A_{ix} + \sum_x A_{xj})} = \frac{|\Gamma_i \cap \Gamma_j|}{\frac{1}{2}(k_i + k_j)} \approx \frac{|\Gamma_i \cap \Gamma_j|}{k_i + k_j}$$

- *hub promoted index* [RSM<sup>+</sup>02] for  $i$  and  $j$  is

$$s_{ij} = \frac{\sum_x A_{ix}A_{xj}}{\min(\sum_x A_{ix}, \sum_x A_{xj})} = \frac{|\Gamma_i \cap \Gamma_j|}{\min(k_i, k_j)}$$

- *hub depressed index* [LZ10] for  $i$  and  $j$  is

$$s_{ij} = \frac{\sum_x A_{ix}A_{xj}}{\max(\sum_x A_{ix}, \sum_x A_{xj})} = \frac{|\Gamma_i \cap \Gamma_j|}{\max(k_i, k_j)}$$

# prediction *models*

*links* predicted by *graph/network models*

- *configuration model index* [LHN06] for  $i$  and  $j$  is

$$s_{ij} = \frac{n \sum_x A_{ix} A_{xj}}{\sum_x A_{ix} \sum_x A_{jx}} = \frac{|\Gamma_i \cap \Gamma_j|}{k_i k_j / n} \approx \frac{|\Gamma_i \cap \Gamma_j|}{k_i k_j}$$

- *preferential attachment index* [BA99] for  $i$  and  $j$  is

$$s_{ij} = \sum_x A_{ix} \sum_x A_{xj} = k_i k_j$$

- *random graph index* [ER59] for  $i$  and  $j$  is

$$s_{ij} = \frac{\langle k \rangle}{n-1} \approx \text{const.}$$

# prediction *dynamics*

*links* predicted by *local dynamics*

- *resource allocation index* [ZLZ09] for  $i$  and  $j$  is

$$s_{ij} = \sum_x \frac{A_{ix}A_{xj}}{\sum_y A_{xy}} = \sum_{x \in \Gamma_i \cap \Gamma_j} \frac{1}{k_x}$$

- *Adamic-Adar similarity index* [AA03] for  $i$  and  $j$  is

$$s_{ij} = \sum_x \frac{A_{ix}A_{xj}}{\log \sum_y A_{xy}} = \sum_{x \in \Gamma_i \cap \Gamma_j} \frac{1}{\log k_x}$$

- *random walk similarity index* [TFP06] for  $i$  and  $j$  is

$$p_i^t = \alpha \sum_{j \in \Gamma_i} p_j^t / k_j + (1 - \alpha) \delta_{it} \quad s_{ij} = p_i^j + p_j^i$$



# prediction *clusters*

## *links* predicted by *node clusters*

- *community structure index* [YG11] for  $i$  and  $j$  is
  - $\{C\}$  *communities* by *Infomap* [RB08] or *Leiden* [TWVE19]
  - $n_i$  and  $m_i$  *number of nodes* and *links* within  $C_i$

$$s_{ij} = \begin{cases} \frac{m_i}{\binom{n_i}{2}} & \text{if } c_i = c_j \\ -\infty & \text{otherwise} \end{cases}$$

- *block model index* [HLL83, DBF05] for  $i$  and  $j$  is
  - $\{C\}$  *clusters* by *stochastic block models* [Pei15]
  - $m_{ij}$  *number of links* between  $C_i$  and  $C_j$

$$s_{ij} = \begin{cases} \frac{m_i}{\binom{n_i}{2}} & \text{if } c_i = c_j \\ \frac{m_{ij}}{n_i n_j} & \text{otherwise} \end{cases}$$

# prediction *framework*

link *prediction* as *ranking problem*

— *standard link prediction* setting

1.  $L_N \leftarrow$  randomly *sample*  $m/10$  *unlinked nodes*  $\{i, j\} \notin L$
2.  $L_P \leftarrow$  *remove* random  $m/10$  *node links*  $\{i, j\} \in L$
3. *compute*  $s_{ij}$  for  $\{i, j\} \in L_N \cup L_P$  on *resulting*  $L$

— *temporal link prediction* setting

1.  $L_N \leftarrow$  randomly *sample*  $|L_P|$  *unlinked nodes*  $\{i, j\} \notin L$
2.  $L_P \leftarrow$  *remove node links*  $\{i, j\} \in L$  *after time*  $t$
3. *compute*  $s_{ij}$  for  $\{i, j\} \in L_N \cup L_P$  on  $L$  *at time*  $t$

— *Pearson/Spearman correlation* or *AUC measure*

$$\left[ \overbrace{0 \ 0 \ 0 \ \dots \ 0 \ 0 \ 0}^{\text{ideal } s_{ij} \text{ for } L_N} \ \overbrace{1 \ 1 \ 1 \ \dots \ 1 \ 1 \ 1}^{\text{ideal } s_{ij} \text{ for } L_P} \right]$$

# prediction *scale-free*

- *link prediction* in synthetic *scale-free graph* [BA99]
  - 1st *highest AUC* by *stochastic block models* [Pei15]
  - 2nd *highest AUC* by *preferential attachment* [BA99]

class	index	Pearson	Spearman	AUC
models	<i>preferential</i>	0.128	0.347	<b>0.701</b>
equivalence	neighbors	0.105	0.135	0.530
	Jaccard	0.019	0.131	0.529
	Salton	0.043	0.131	0.529
	Leicht	-0.008	0.131	0.529
dynamics	allocation	0.091	0.135	0.530
	Adamic-Adar	0.104	0.135	0.530
clusters	modularity	0.002	0.005	0.502
	map equation	0.009	0.034	0.503
	<i>block model</i>	0.168	0.370	<b>0.711</b>
baseline	random	-0.001	-0.001	0.499

## prediction *small-world*

- *link prediction* in synthetic *small-world graph* [WS98]
  - 1st *highest AUC* by *stochastic block models* [Pei15]
  - 2nd *highest AUC* by *common neighbors index* [LW71]

class	index	Pearson	Spearman	AUC
models	preferential	-0.563	-0.547	0.187
equivalence	<i>neighbors</i>	0.721	0.786	<b>0.903</b>
	<i>Jaccard</i>	0.686	0.785	<b>0.903</b>
	<i>Salton</i>	0.729	0.785	<b>0.902</b>
	<i>Leicht</i>	0.730	0.785	<b>0.903</b>
dynamics	<i>allocation</i>	0.719	0.785	<b>0.902</b>
	<i>Adamic-Adar</i>	0.720	0.785	<b>0.902</b>
clusters	modularity	0.743	0.754	0.885
	map equation	0.643	0.649	0.807
	<i>block model</i>	0.737	0.754	<b>0.931</b>
baseline	random	-0.003	-0.002	0.499

# prediction *human*

— *link prediction* in *human protein interaction* map

1st *highest AUC* by *stochastic block models* [Pei15]

2nd *highest AUC* by *preferential attachment* [BA99]

class	index	Pearson	Spearman	AUC
models	<i>preferential</i>	0.231	0.719	<b>0.915</b>
equivalence	neighbors	0.342	0.676	0.845
	Jaccard	0.301	0.648	0.830
	Salton	0.391	0.646	0.830
	Leicht	-0.005	0.625	0.819
dynamics	allocation	0.291	0.681	0.847
	Adamic-Adar	0.343	0.680	0.847
clusters	modularity	0.284	0.381	0.672
	map equation	0.220	0.408	0.660
	<i>block model</i>	0.344	0.746	<b>0.929</b>
baseline	random	0.000	0.000	0.500

## prediction *P2P*

- *link prediction* in *P2P file transfer* network [LKF07]
  - 1st *highest AUC* by *stochastic block models* [Pei15]
  - 2nd *highest AUC* by *preferential attachment* [BA99]

class	index	Pearson	Spearman	AUC
models	<i>preferential</i>	0.379	0.378	<i>0.717</i>
equivalence	neighbors	0.113	0.120	0.515
	Jaccard	0.093	0.120	0.515
	Salton	0.098	0.120	0.515
	Leicht	0.055	0.120	0.515
dynamics	allocation	0.087	0.120	0.515
	Adamic-Adar	0.102	0.120	0.515
clusters	modularity	0.081	0.121	0.531
	map equation	0.096	0.113	0.513
	<i>block model</i>	0.487	0.621	<i>0.837</i>
baseline	random	-0.002	-0.002	0.499

# prediction *IMDb*

- *link prediction* in *IMDb collaboration* network [BA99]
  - 1st *highest AUC* by *stochastic block models* [Pei15]
  - 2nd *highest AUC* by *resource allocation index* [ZLZ09]

class	index	Pearson	Spearman	AUC
models	preferential	0.359	0.589	0.840
equivalence	<i>neighbors</i>	0.491	0.875	<i>0.970</i>
	<i>Jaccard</i>	0.609	0.876	<i>0.970</i>
	<i>Salton</i>	0.724	0.877	<i>0.970</i>
	<i>Leicht</i>	0.355	0.869	<i>0.967</i>
dynamics	<i>allocation</i>	0.627	0.878	<i>0.971</i>
	<i>Adamic-Adar</i>	0.520	0.876	<i>0.970</i>
clusters	modularity	0.345	0.826	0.948
	map equation	0.421	0.785	0.909
	<i>block model</i>	0.544	0.856	<i>0.986</i>
baseline	random	-0.003	-0.003	0.498

# prediction *nd.edu*

- *link prediction* in *nd.edu web* graph [BA99]
  - 1st *highest AUC* by *modularity optimization* [BGLL08]
  - 2nd *highest AUC* by *resource allocation index* [ZLZ09]

class	index	Pearson	Spearman	AUC
models	preferential	0.094	0.548	0.816
equivalence	<i>neighbors</i>	0.346	0.717	<i>0.855</i>
	<i>Jaccard</i>	0.453	0.716	<i>0.854</i>
	<i>Salton</i>	0.526	0.716	<i>0.854</i>
	<i>Leicht</i>	0.257	0.715	<i>0.854</i>
dynamics	<i>allocation</i>	0.181	0.718	<i>0.855</i>
	<i>Adamic-Adar</i>	0.334	0.718	<i>0.855</i>
clusters	<i>modularity</i>	0.197	0.767	<i>0.893</i>
	map equation	0.391	0.703	0.844
	block model	-	-	-
baseline	random	-0.001	-0.001	0.499



# prediction WoS

- *link prediction* in *WoS citation* network [ŠF17]
  - 1st *highest AUC* by *modularity optimization* [BGLL08]
  - 2nd *highest AUC* by *common neighbors index* [LW71]

class	index	Pearson	Spearman	AUC
models	preferential	0.082	0.509	0.794
equivalence	<i>neighbors</i>	0.434	0.754	<b>0.880</b>
	<i>Jaccard</i>	0.499	0.753	<b>0.880</b>
	<i>Salton</i>	0.574	0.753	<b>0.880</b>
	<i>Leicht</i>	0.258	0.753	<b>0.880</b>
dynamics	<i>allocation</i>	0.449	0.754	<b>0.880</b>
	<i>Adamic-Adar</i>	0.454	0.754	<b>0.880</b>
clusters	<i>modularity</i>	0.082	0.779	<b>0.908</b>
	map equation	0.392	0.546	0.734
	block model	-	-	-
baseline	random	0.000	0.000	0.500

# prediction *Texas*

- *link prediction* in *Texas road map* [LLDM09]
  - 1st *highest AUC* by *modularity optimization* [BGLL08]
  - 2nd *highest AUC* by *map equation method* [RB08]

class	index	Pearson	Spearman	AUC
models	<i>preferential</i>	-0.353	-0.311	<i>0.322</i>
equivalence	neighbors	0.230	0.233	0.551
	Jaccard	0.217	0.232	0.551
	Salton	0.225	0.232	0.551
	Leicht	0.202	0.232	0.551
dynamics	allocation	0.225	0.232	0.551
	Adamic-Adar	0.225	0.232	0.551
clusters	<i>modularity</i>	0.060	0.736	<i>0.868</i>
	<i>map equation</i>	0.335	0.362	<i>0.616</i>
	block model	-	-	-
baseline	random	0.000	0.000	0.500

# inference *references*



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