

link *bridging*

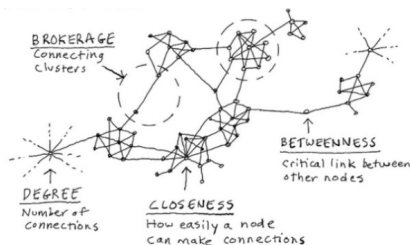
introduction to *network analysis* (*ina*)

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bridging *measures*

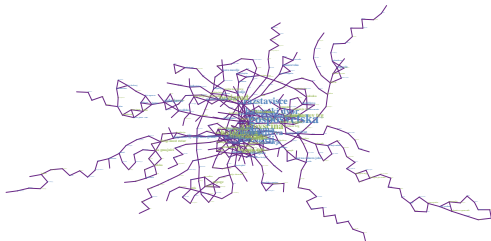
which *links* are most *important*?

- *link bridging measures* for (un)directed networks
 - *betweenness-based* centrality [Fre77, FBW91, New05]
- *link embeddedness measures* for (un)directed networks
 - *topological overlap* measures [RSM⁺02, OSH⁺07, dNMB11]



networkology *LPP*

- partial *LPP public bus transport network**
- $n = 416$ bus stops with $\langle k \rangle = 2.72$ connections
- *giant component* 95.4% nodes (6 components)
- “*small-world*” with $\langle C \rangle = 0.09$ and $\langle d \rangle = 14.26$
- “*scale-free*” with $\gamma = 2.43$ for cutoff $k_{min} = 2$



* reduced to largest connected component of simple undirected graph

bridging *betweenness*

important *links* are *between other nodes*

- for (*un*)*directed* G *link betweenness* σ [Fre77] of $\{i, j\}$ is
 - g_{st} is number of *geodesic paths between* s and t
 - g_{st}^{ij} is number of *such geodesic paths through* $\{i, j\}$

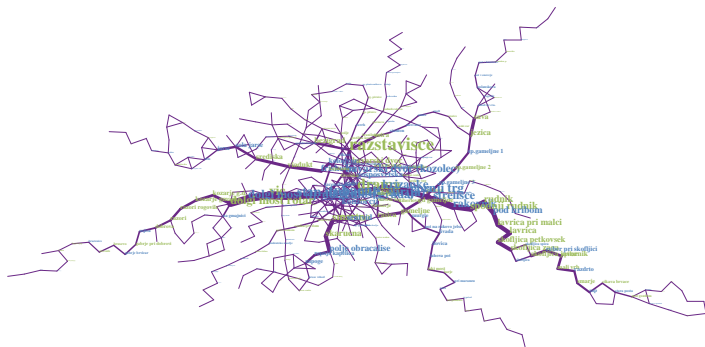
$$\sigma_{ij} = \sum_{st \notin \{i, j\}} \frac{g_{st}^{ij}}{g_{st}}$$

- σ considers *only geodesic paths* [FBW91, New05]



networkology *betweenness*

- *link betweenness* σ in partial LPP network[†]
- *highest* $\sigma_{ij} = 0.176n^2$ link is {*Vič, Stan in dom*}



[†] reduced to largest connected component of simple undirected graph

bridging *bridgeness*

important *links* are *bridges between nodes*

- for (un)directed G *link bridgeness* $\tilde{\sigma}$ [JMK⁺16] of $\{i, j\}$ is
 - g_{st} is number of *geodesic paths between s and t*
 - g_{st}^{ij} is number of *such geodesic paths through $\{i, j\}$*

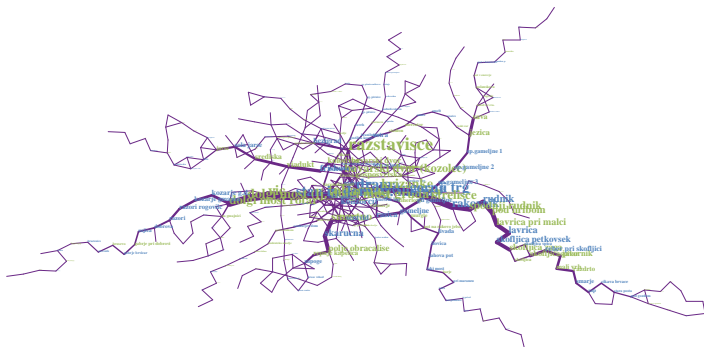
$$\tilde{\sigma}_{ij} = \sigma_{ij} - \sum_{st \in \Gamma_i \cup \Gamma_j} \frac{g_{st}^{ij}}{g_{st}} = \sum_{st \notin \Gamma_i \cup \Gamma_j} \frac{g_{st}^{ij}}{g_{st}}$$

- σ mixes *local centers* with *global bridges* [JMK⁺16]



networkology *bridgeness*

- *link bridgeness* $\tilde{\sigma}$ in partial LPP network[‡]
- *highest* $\tilde{\sigma}_{ij} = 0.169n^2$ link is {*Vič, Stan in dom*}



[‡] reduced to largest connected component of simple undirected graph

bridging *embeddedness*

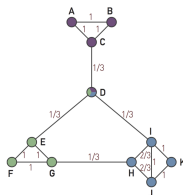
important *links* are *embedded between nodes*

- for *undirected* G *link embeddedness*[§] θ [OSH⁺07] of $\{i, j\}$ is
 - Γ_i is set of *neighbors* or *neighborhood* of i

$$\theta_{ij} = \frac{|\Gamma_i \cap \Gamma_j|}{k_i - 1 + k_j - 1 - |\Gamma_i \cap \Gamma_j|} \quad \theta_{ij} = 0 \text{ for } k_i = k_j = 1$$

- μ -*corrected link embeddedness* $\tilde{\theta}$ [Bat19] of $\{i, j\}$ is
 - μ is *maximum* number of *triangles* over *links*

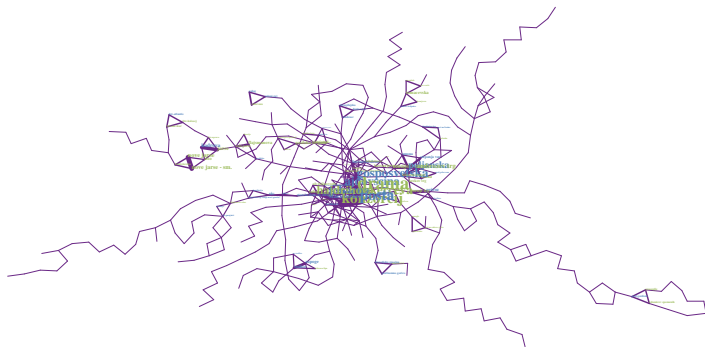
$$\tilde{\theta}_{ij} = \frac{|\Gamma_i \cap \Gamma_j|}{\mu + \max(k_i, k_j) - 1 - |\Gamma_i \cap \Gamma_j|}$$



[§] θ & $\tilde{\theta}$ better known as topological overlap indices/weights

networkology μ -embeddedness

- μ -corrected embeddedness $\tilde{\theta}$ in partial LPP network^{||}
- highest $\tilde{\theta}_{ij} = 0.4$ links are {*Pošta*, *Konzorcij*} etc.



^{||} reduced to largest connected component of simple undirected graph

bridging *overview*

which *links* are most *important*?

1 IA												18 VIIIA												
1	DC																						2	EC
	Degree Centrality																							Eigenvector Centrality
2	BC	CC											13 IIIA	14 IVA	15 VA	16 VIA	17 VIIA						3	PR
	Betweenness Centrality	Closeness Centrality											SC	C _{COEF}	C _{COEF} ⁻¹	MNC	EC _{COEF}							PageRank
3	RL	IC											CC _{COEF}	SC ₀	LAC	DMNC	SEC _{COEF}						4	LR
	Range-Linked Betweenness	Information Centrality											old Subgraph Centrality	old Subgraph Centrality	loc. avg. Connectivity	dens. max. length comp.	sum of EC _{COEF}							LeaderRank
4	BN	RC	IG	DC _{0x}	BC _{0x}	CC _{0x}	EC _{0x}	KS _{0x}	PR _{0x}	IG _{0x}	RC _{0x}	DC _{0x}	BC _{0x}	SC ₁	KL	COC _{COEF}	PEC _{COEF}					5	KS	
	Betweenness Centrality	Radiality Centrality	Integration											new Subgraph Centrality	Clique Level	cooper. weight CO _{COEF}	PCC + EC _{COEF}						KatzRank	
5	RWBC	RWCC	CC _{2,3,4}	EC _{0x0x}	PR _{0x0x}	KS _{0x0x}	CC _{0x0x}	RC _{0x0x}	IG _{0x0x}	DC _{0x0x}	BC _{0x0x}	CC _{0x0x}	KS _{0x0x}	SC ₂	β	SC ₂	NC					6	EC ₃	
	RandomWalk Betweenness	RandomWalk Closeness	2,3,4-localized CC											Bipartivity	2-localized SC	Neighborhood Centrality	2-localized EC							
6	σ	ECC	WDC	DC _{0x0x0x}	CC _{0x0x0x}	KS _{0x0x0x}	PR _{0x0x0x}	IG _{0x0x0x}	DC _{0x0x0x}	BC _{0x0x0x}	CC _{0x0x0x}	KS _{0x0x0x}	SC ₃	LI								7	EC ₄	
	sigma Centrality	Eccentricity	Weighted Degree											3-localized SC	Lobby Index		4-localized EC							
7	BC _{2,3,4}	ECC ⁻¹	SDC	DC _{0x0x0x0x}	CC _{0x0x0x0x}	KS _{0x0x0x0x}	PR _{0x0x0x0x}	IG _{0x0x0x0x}	DC _{0x0x0x0x}	BC _{0x0x0x0x}	CC _{0x0x0x0x}	KS _{0x0x0x0x}	SC ₄											
	2,3,4-localized-BC	Inverse Eccentricity	Sphere Degree Centrality											4-localized SC										

Z

mass

C

Name

Hybrid

22	FC	23	FD	24	US	25	DIS	26	ASS	27	DAM	117	UC
	Functional Centrality		Functional Diversity		UniScore		Pairwise Dis-connectivity		Assortative Mixing		Damage		United compl. Centrality

28	EI	29	CM	30	NαC	31	MC	13	HGI	116	HYP	118	HC
	Essentiality Index		Complexity Measure		Normalized α Centrality		Modular Centrality		Hungry Graph Information		Hyperbolic Index		Harmonic Centrality

Betweenness-based

Distance-based

Linear Combinations

Subgraph-based

Clustering Coefficient-based

Edge Clustering Coefficient-based

Spectral-based

Miscellaneous

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bridging *references*



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bridging *references*



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