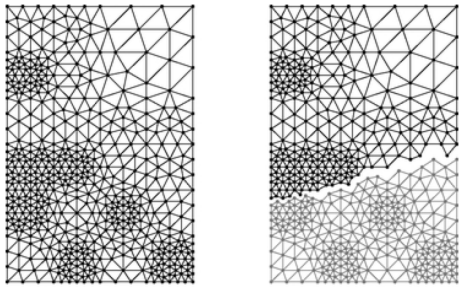
network *clustering*

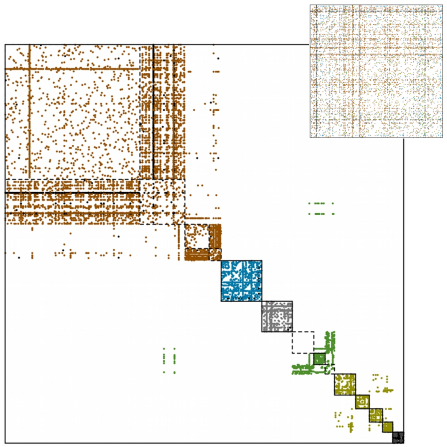
introduction to *network analysis* (*ina*)

Lovro Šubelj
University of Ljubljana
spring 2023/24

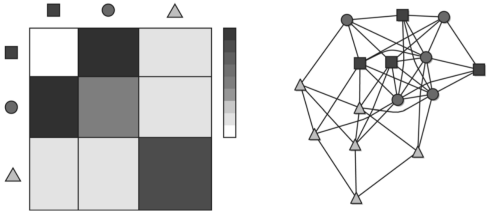
clustering *overview*



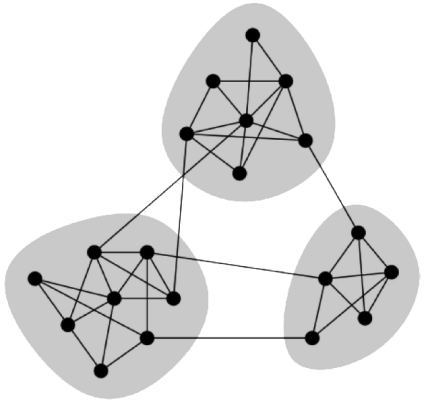
graph partitioning [KL70, Fie73]



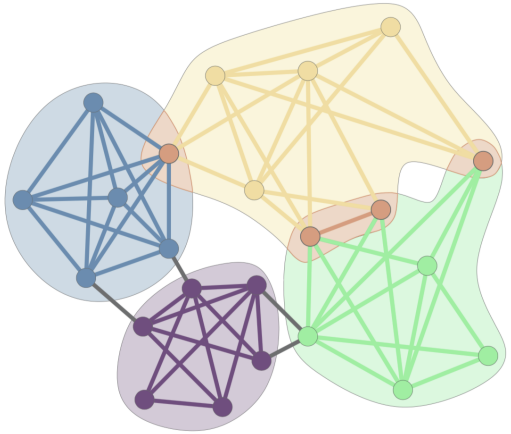
blockmodeling [LW71, WR83]



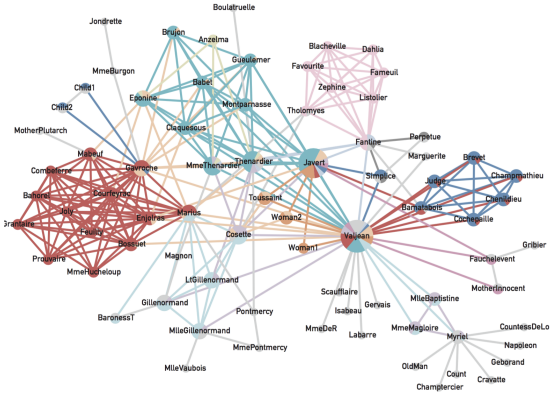
stochastic block models [Pei15]



communities [GN02]

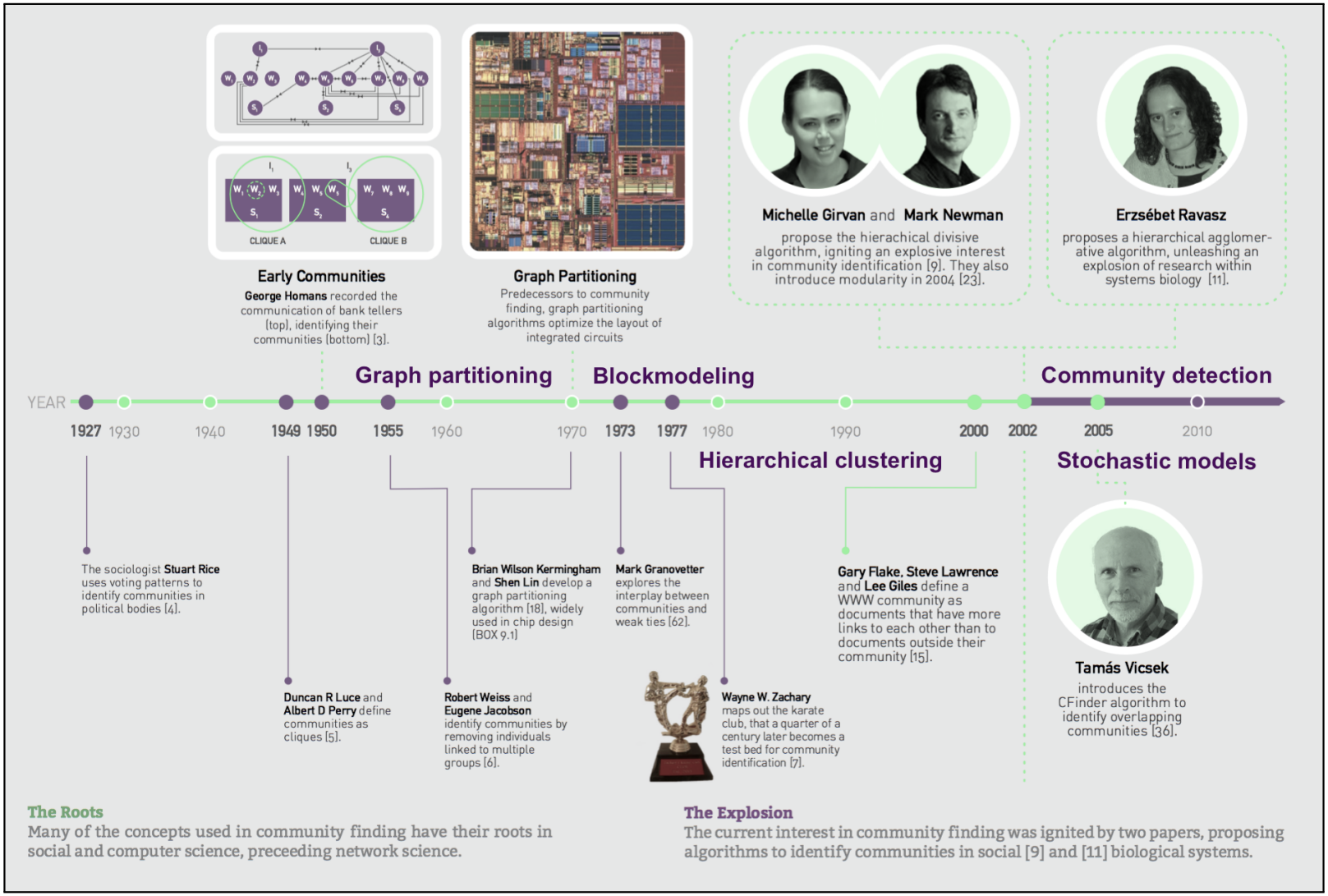


overlapping communities [PDFV05]



link communities [EL09, ABL10]

clustering *history*



graph *partitioning*

introduction to *network analysis* (*ina*)

Lovro Šubelj
University of Ljubljana
spring 2023/24

partitioning *bisection*

— Kernighan-Lin *graph bisection* [KL70]

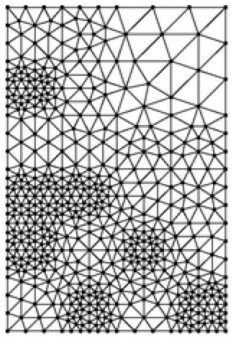
– define *bisection quality* as *cut size*

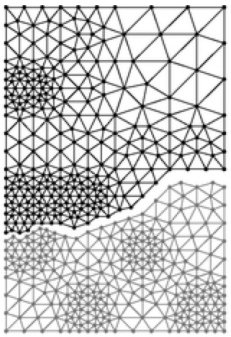
$$R = \frac{1}{2}\sum_{ij} A_{ij}(1-\delta_{c_i c_j}) \qquad \forall i : c_i = \pm 1$$

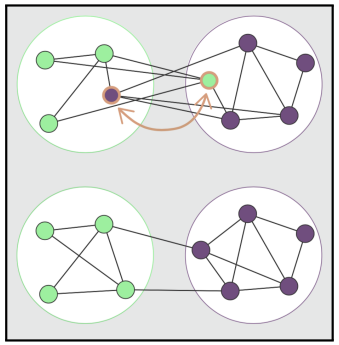
1. *swap nodes* by *minimizing cut size* $\mathcal{O}(cn^2m)$

$$\Delta R_{ij} = k_i^{ext} - k_i^{in} + k_j^{ext} - k_j^{in} - 2A_{ij}$$

2. repeat 1. *until* $\min(n_1, n_2)$ *nodes swapped*
3. return *bisection minimizing cut size*







*example mesh bisection with cut size equal to 40

partitioning *spectral*

— Fiedler *graph bisection* [Fie73]

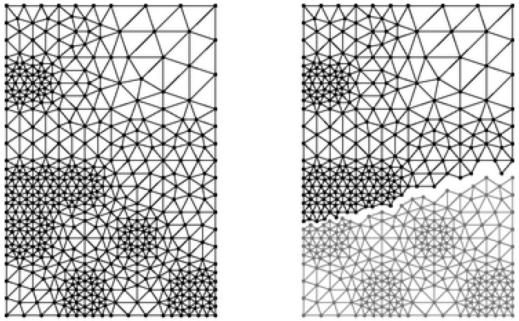
- define *bisection quality* as *cut size*

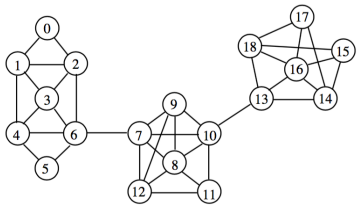
$$R = \frac{1}{4}\sum_{ij} A_{ij}(1 - s_i s_j) \qquad \forall i : s_i = \delta_{c_i c_1} - \delta_{c_i c_2}$$

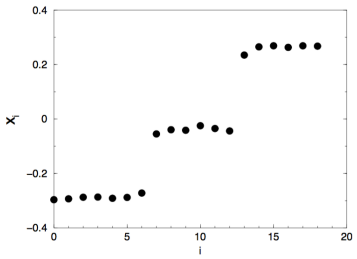
- formulate *eigenvector problem* of *graph Laplacian*

$$R = \frac{1}{4}\sum_i k_i s_i^2 - \frac{1}{4}\sum_{ij} A_{ij} s_i s_j = \frac{1}{4}\sum_{ij}(k_i \delta_{ij} - A_{ij}) s_i s_j = \frac{1}{4} s^T L s \simeq \frac{1}{4} v^T L v = \frac{n_1 n_2}{n}\lambda$$

1. find *eigenvector* v_2 *of* L with *algebraic connectivity* λ_2 $\mathcal{O}(nm)$
2. assign n_1 *nodes* with *largest/smallest* v_2 to C_1
3. return *bisection minimizing cut size*







see graclus and metis implementations

† example mesh bisection with cut size equal to 46

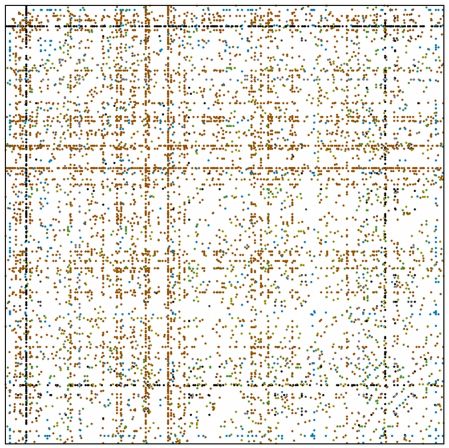
partitioning *blockmodeling*

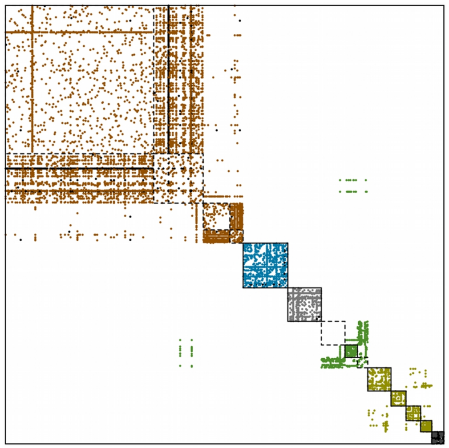
— standard *equivalence blockmodeling* [DBF05]

- define *node similarity* as *(structural) equivalence*

$$s_{ij} \sim |\Gamma_i \cap \Gamma_j|$$

1. *blockmodeling* by *(hierarchical) clustering* $\mathcal{O}(n^2)$
2. return *block model* at desired *clustering resolution*





see catrege implementation

‡ `javax.swing`, `javax.management`, `javax.naming`, `javax.print`, `javax.xml`, `javax.lang` etc.

community detection

introduction to *network analysis* (*ina*)

Lovro Šubelj
University of Ljubljana
spring 2023/24

community *agglomerative*

— Ravasz *hierarchical clustering* [RSM+02]

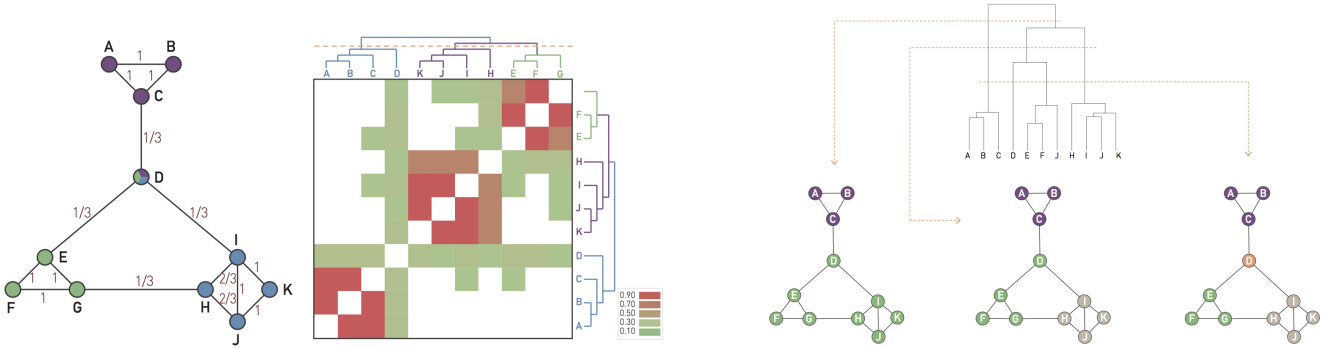
- define *node similarity* as *topological overlap*

$$s_{ij} = \frac{|\Gamma_i \cap \Gamma_j| + A_{ij}}{\min(k_i, k_j)}$$

- define *cluster similarity* as *average linkage*

$$S_{ij} = \frac{1}{n_i n_j} \sum_{xy} s_{xy} \delta_{c_x c_i} \delta_{c_y c_j}$$

1. bottom-up *agglomerative hierarchical clustering* $\mathcal{O}(n^2)$
2. cut *cluster dendrogram* at desired *clustering resolution*



community *divisive*

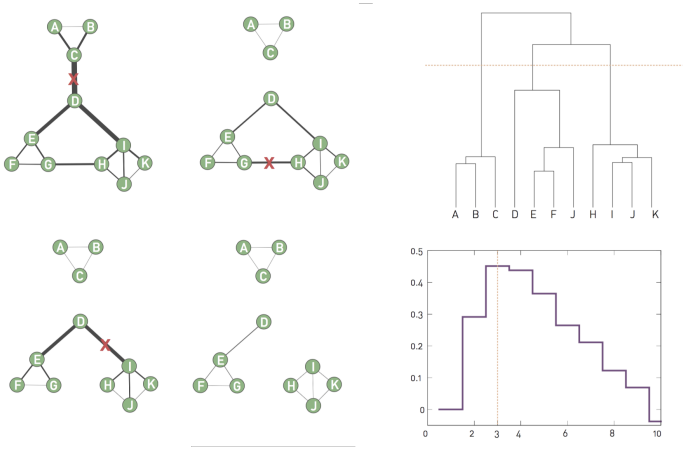
— Girvan-Newman *hierarchical clustering* [GN02]

– define *node dissimilarity* as *link betweenness*

$$\sigma_{ij} = \sum_{st \notin \{i,j\}} \frac{g_{st}^{ij}}{g_{st}}$$

1. top-down *divisive hierarchical clustering* $\mathcal{O}(nm^2)$
2. cut *cluster dendrogram* at *maximum modularity*

$$Q = \frac{1}{2m} \sum_{ij} (A_{ij} - \frac{k_i k_j}{2m}) \delta_{c_i c_j}$$

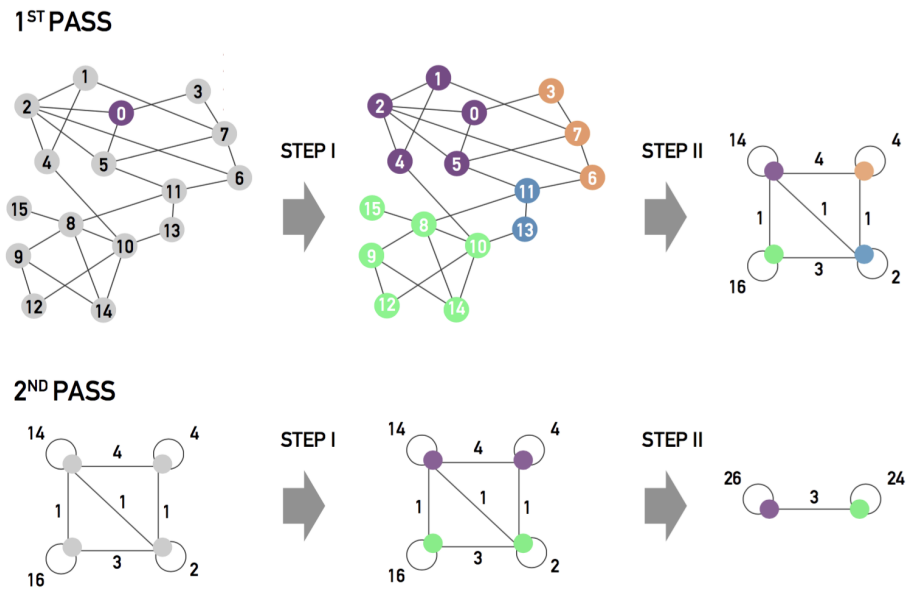


community *modularity*

— Louvain *modularity optimization* [BGLL08]

1. set *node community* by *modularity optimization* $\mathcal{O}(cm)$
2. *aggregate community nodes into supernodes* and repeat 1.
3. return *community structure maximizing modularity*

$$Q = \frac{1}{2m} \sum_{ij} (A_{ij} - \frac{k_i k_j}{2m}) \delta_{c_i c_j}$$

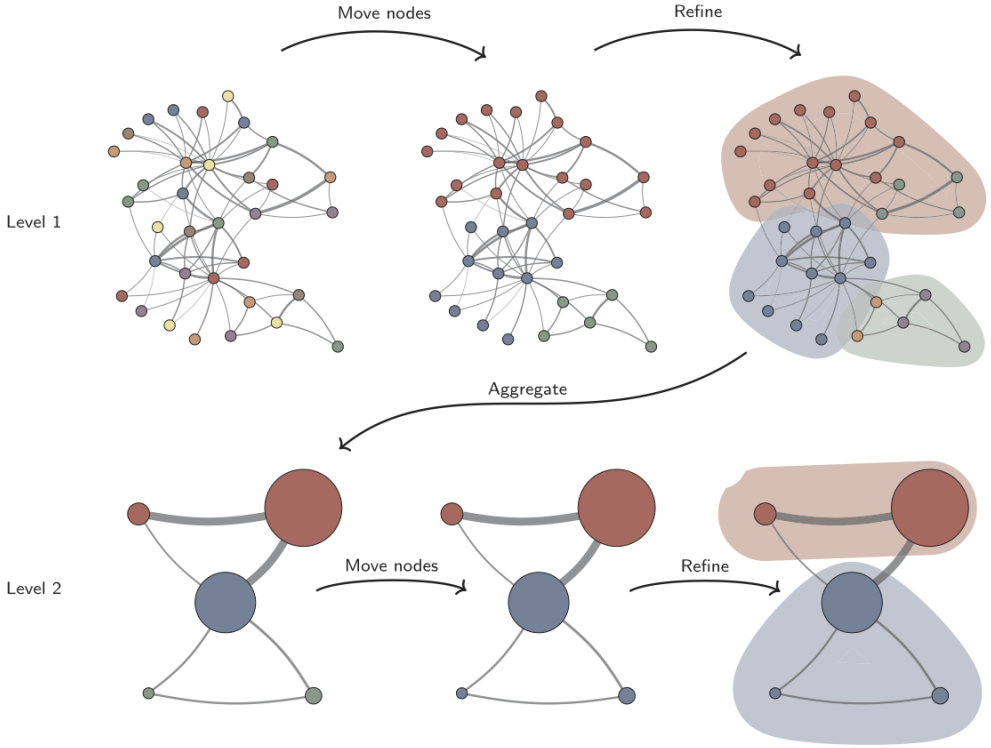


see louvain implementation

community *modularity*

— Leiden *modularity optimization* [TWVE19]

x. *improved* Louvain *algorithm* with *quality guarantees*



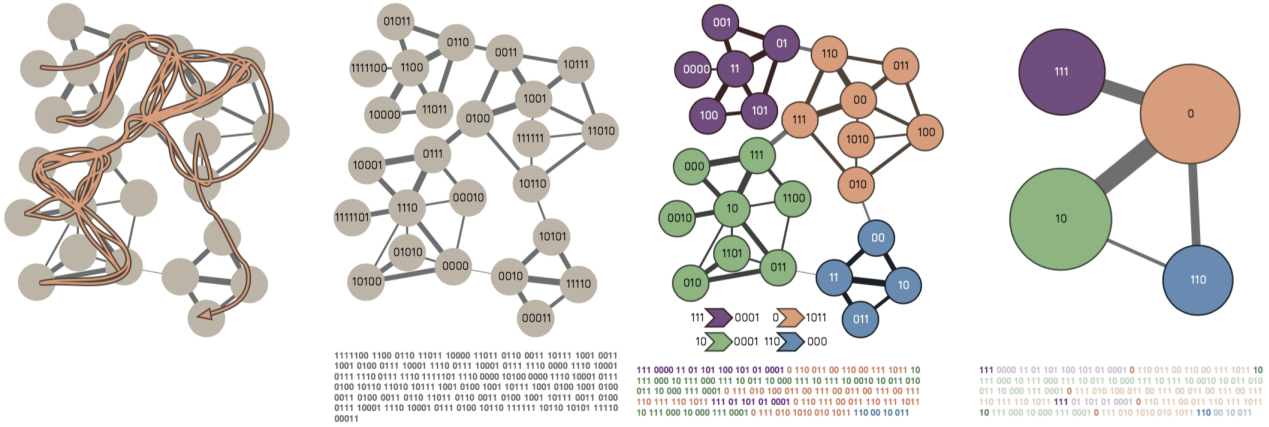
see leidenalg implementation

community *map equation*

— Infomap *map equation compression* [RB08]

1. set *node community* by *optimal coding* $\mathcal{O}(m \log m)$
2. *compress community nodes into supernodes* and repeat 1.
3. return *community structure maximizing map equation*

$$\mathcal{L} = \sum_i p_{i\rightsquigarrow} H(\widetilde{\mathcal{C}}) + \sum_i p_{i\hookleftarrow} H(\mathcal{C}_i)$$



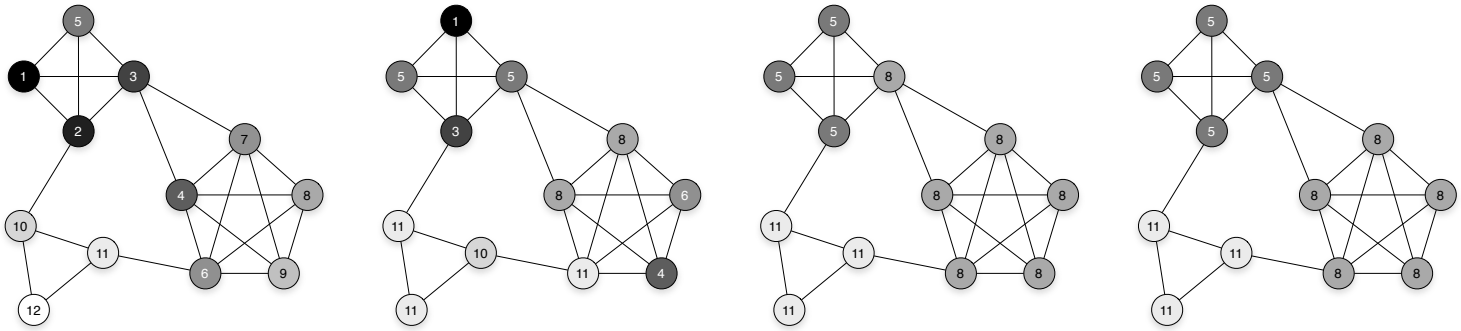
see mapequation implementation

community *propagation*

— Raghavan *label propagation* [RAK07, Šub20, TŠ23]

1. set *node community* by *neighbors frequency* $\mathcal{O}(cm)$
2. *randomly shuffle nodes* and repeat 1. *until convergence*
3. return *community structure connected components*

$$\forall i : c_i = \arg\max_c \sum_j A_{ij} \delta_{c_j c}$$



see networkx implementation

community *percolation*

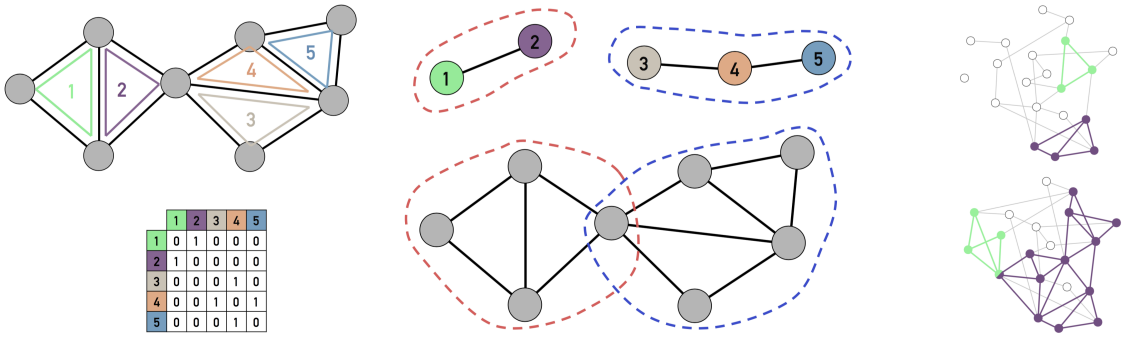
— Palla *clique percolation* [PDFV05, KKKS08]

1. find *k-node cliques* by *sequential enumeration* $\mathcal{O}(n_k)$
2. *merge clique nodes into supernodes* and *link adjacent*

   adjacent k-node cliques share $k-1$ nodes

3. return *clique structure connected components*

   clique percolation at $(kn-n)^{\frac{1}{1-k}}$



see kclique implementation

community *links*

— Ahn *link clustering* [EL09, ABL10]

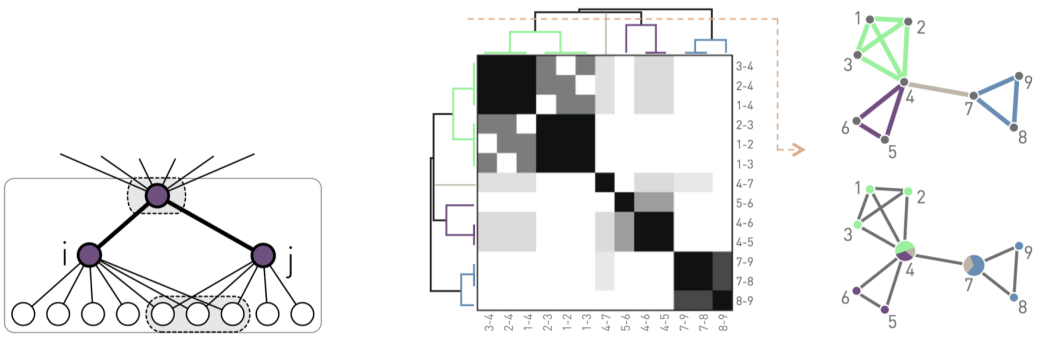
- define *link similarity* as *neighbors index*

$$\forall ij \in \Gamma_x : s_{ij}^x = \frac{|\Gamma_i^+ \cap \Gamma_j^+|}{|\Gamma_i^+ \cup \Gamma_j^+|}$$

- define *cluster similarity* as *single linkage*

$$S_{ij} = \max_{xy \in \Gamma_z} (s_{xy}^z \delta_{c_{xz} c_i} \delta_{c_{yz} c_j})$$

1. bottom-up *agglomerative hierarchical clustering* $\mathcal{O}(m^2)$
2. cut *cluster dendrogram* at desired *clustering resolution*



see linkcomm implementation

community *measures*

— *degree* K, *expansion* E and *Flake* F [FLG00, RCC+04] of $\{C\}$

$$K = \frac{1}{n}\sum_{ij} A_{ij}\delta_{c_i c_j} = \langle k \rangle - E \qquad F = \frac{|\{i : \sum_j A_{ij}\delta_{c_i c_j} < k_i/2\}|}{n}$$

— *normalized mutual information* NMI [DDGDA05] of $\{C\}$,$\{D\}$

- p_c & p_{cd} are *standard* & *joint distributions* of $\{C\}$,$\{D\}$
- $H(\mathcal{C})$ & $H(\mathcal{C}|\mathcal{D})$ are *standard* & *conditional entropies*
- MI & VI are *mutual* & *variation* of *information*

$$NMI = \frac{2MI(\mathcal{C},\mathcal{D})}{H(\mathcal{C})+H(\mathcal{D})} = \frac{2H(\mathcal{C})-2H(\mathcal{C}|\mathcal{D})}{H(\mathcal{C})+H(\mathcal{D})} = \frac{2H(\mathcal{C})+2\sum_{CD} p_{cd}\log\frac{p_{cd}}{p_d}}{-\sum_C p_c \log p_c + H(\mathcal{D})}$$

— *normalized variation of information* NVI [Mei07, KLN08]

$$NVI = \frac{VI(\mathcal{C},\mathcal{D})}{\log n} = \frac{H(\mathcal{C}|\mathcal{D})+H(\mathcal{D}|\mathcal{C})}{\log n}$$

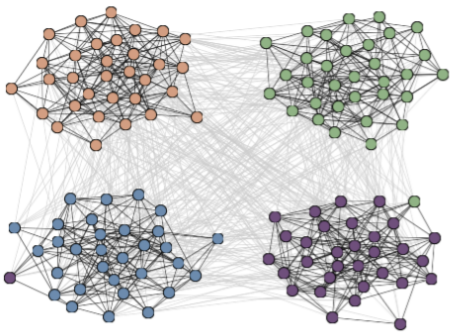
— other measures include *adjusted rand index* ARI etc.

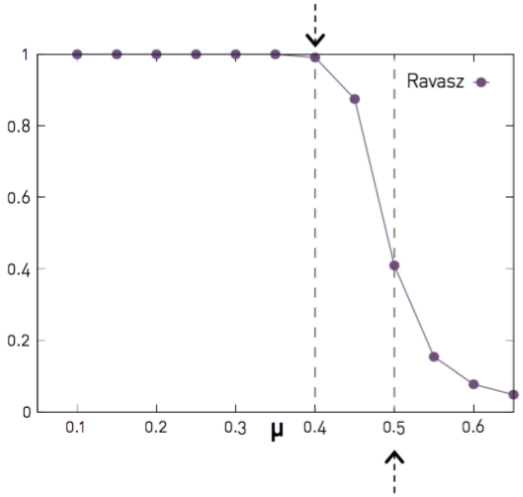
community *benchmarks*

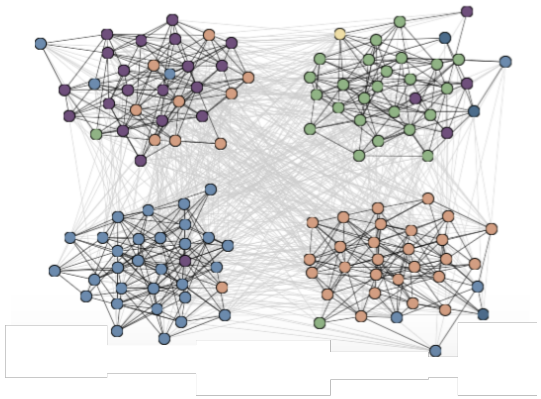
— Girvan-Newman *synthetic graphs* [GN02]

— *planted partition* controlled by *mixing parameter* μ

$n = 128$ $\langle k \rangle = \langle k^{int} \rangle + \langle k^{ext} \rangle = 16$ $\mu = \frac{\langle k^{ext} \rangle}{\langle k \rangle}$



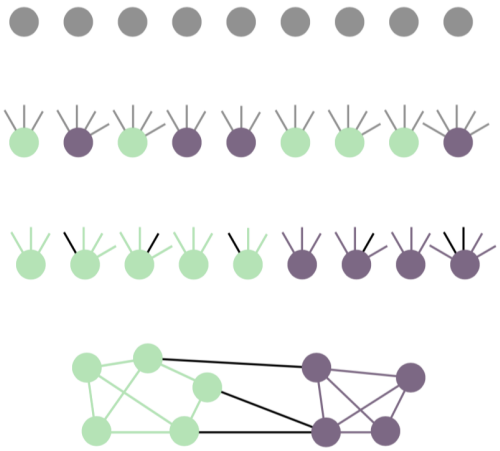


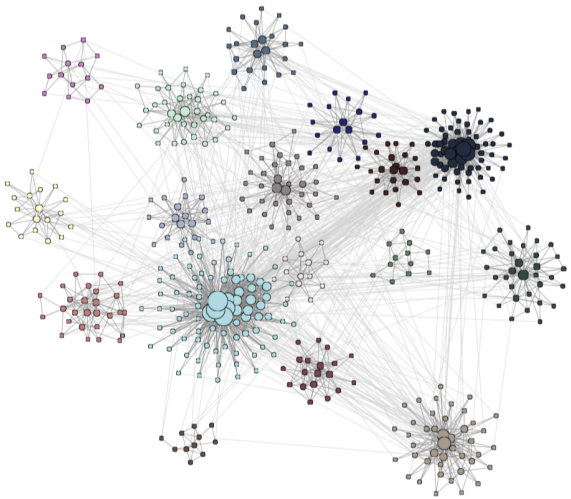


community *benchmarks*

- Lanchichinetti *synthetic graphs* [LFR08]
- *power-law distributions* $p_k \sim k^{-\gamma_k}$ & $p_s \sim s^{-\gamma_s}$
- *planted communities* controlled by *mixing parameter* μ

$n = 1000,\ n_c \in [10, 50]$ $\qquad$ $\gamma_k \in [2, 3],\ \gamma_s \in [1, 2]$ $\qquad$ $\mu = \frac{\langle k^{ext} \rangle}{\langle k \rangle}$





clustering *references*

Yong-Yeol Ahn, James P. Bagrow, and Sune Lehmann.
Link communities reveal multiscale complexity in networks.
Nature, 466(7307):761–764, 2010.

A.-L. Barabási.
Network Science.
Cambridge University Press, Cambridge, 2016.

V. D. Blondel, J.-L. Guillaume, R. Lambiotte, and E. Lefebvre.
Fast unfolding of communities in large networks.
J. Stat. Mech., P10008, 2008.

Patrick Doreian, Vladimir Batagelj, and Anuska Ferligoj.
Generalized Blockmodeling.
Cambridge University Press, Cambridge, 2005.

Leon Danon, Albert Díaz-Guilera, Jordi Duch, and Alex Arenas.
Comparing community structure identification.
J. Stat. Mech., page P09008, 2005.

Wouter de Nooy, Andrej Mrvar, and Vladimir Batagelj.
Exploratory Social Network Analysis with Pajek: Expanded and Revised Second Edition.
Cambridge University Press, Cambridge, 2011.

David Easley and Jon Kleinberg.
Networks, Crowds, and Markets: Reasoning About a Highly Connected World.
Cambridge University Press, Cambridge, 2010.

Ernesto Estrada and Philip A. Knight.
A First Course in Network Theory.
Oxford University Press, 2015.

clustering *references*

T. S. Evans and R. Lambiotte.
Line graphs, link partitions and overlapping communities.
Phys. Rev. E, 80(1):016105, 2009.

M. Fiedler.
Algebraic connectivity of graphs.
Czech. Math. J., 23:298–305, 1973.

Gary William Flake, Steve Lawrence, and C. Lee Giles.
Efficient identification of web communities.
In *Proceedings of the ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, pages 150–160, Boston, MA, USA, 2000.

M. Girvan and M. E. J Newman.
Community structure in social and biological networks.
P. Natl. Acad. Sci. USA, 99(12):7821–7826, 2002.

Jussi M. Kumpula, Mikko Kivelä, Kimmo Kaski, and Jari Saramäki.
Sequential algorithm for fast clique percolation.
Phys. Rev. E, 78(2):026109, 2008.

Brian W. Kernighan and S. Lin.
An efficient heuristic procedure for partitioning graphs.
Bell Sys. Tech. J., 49(2):291–308, 1970.

Brian Karrer, Elizaveta Levina, and M. E. J. Newman.
Robustness of community structure in networks.
Phys. Rev. E, 77(4):046119, 2008.

clustering *references*

Andrea Lancichinetti, Santo Fortunato, and Filippo Radicchi.
Benchmark graphs for testing community detection algorithms.
Phys. Rev. E, 78(4):046110, 2008.

F. Lorrain and H. C. White.
Structural equivalence of individuals in social networks.
J. Math. Sociol., 1(1):49–80, 1971.

Marina Meila.
Comparing clusterings: An information based distance.
J. Multivariate Anal., 98(5):873–895, 2007.

Mark E. J. Newman.
Networks.
Oxford University Press, Oxford, 2nd edition, 2018.

Gergely Palla, Imre Derényi, Illes Farkas, and Tamas Vicsek.
Uncovering the overlapping community structure of complex networks in nature and society.
Nature, 435(7043):814–818, 2005.

Tiago P. Peixoto.
Model selection and hypothesis testing for large-scale network models with overlapping groups.
Phys. Rev. X, 5(1):011033, 2015.

Usha Nandini Raghavan, Reka Albert, and Soundar Kumara.
Near linear time algorithm to detect community structures in large-scale networks.
Phys. Rev. E, 76(3):036106, 2007.

M. Rosvall and C. T. Bergstrom.
Maps of random walks on complex networks reveal community structure.
P. Natl. Acad. Sci. USA, 105(4):1118–1123, 2008.

clustering *references*

Filippo Radicchi, Claudio Castellano, Federico Cecconi, Vittorio Loreto, and Domenico Parisi.
Defining and identifying communities in networks.
P. Natl. Acad. Sci. USA, 101(9):2658–2663, 2004.

E. Ravasz, A. L. Somera, D. A. Mongru, Z. N. Oltvai, and Albert László Barabási.
Hierarchical organization of modularity in metabolic networks.
Science, 297(5586):1551–1555, 2002.

Lovro Šubelj.
Label propagation for clustering.
In Patrick Doreian, Vladimir Batagelj, and Anuška Ferligoj, editors, *Advances in Network Clustering and Blockmodeling*, Computational and Quantitative Social Science, pages 121–150. Wiley, New York, 1st edition, 2020.

Vincent A. Traag and Lovro Šubelj.
Large network community detection by fast label propagation.
Sci. Rep., 13:2701, 2023.

V. A. Traag, Ludo Waltman, and Nees Jan Van Eck.
From Louvain to Leiden: Guaranteeing well-connected communities.
Sci. Rep., 9:5233, 2019.

D. R. White and K. P. Reitz.
Graph and semigroup homomorphisms on networks of relations.
Soc. Networks, 5(2):193–234, 1983.