# network *blockmodeling*

introduction to *network analysis* (*ina*)

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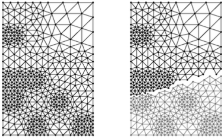
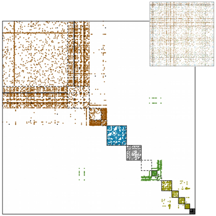
spring 2023/24

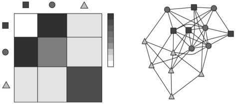
# blockmodeling *overview*



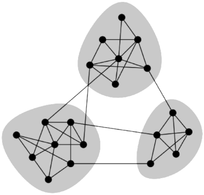
*graph partitioning* [KL70, Fie73]



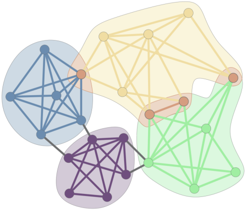
*blockmodeling* [LW71, WR83]



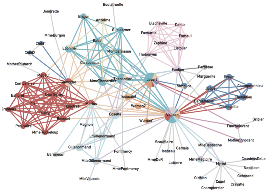
*stochastic block model* [Pei15]



*communities* [GN02]



*overlapping communities* [PDFV05]



*link communities* [EL09, ABL10]

* assortative & disassortative equivalence blockmodeling

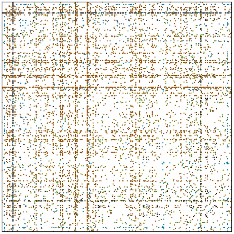
# blockmodeling *equivalence*

— *standard equivalence blockmodeling* [DBF05]

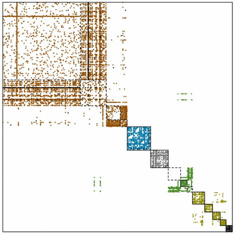
– define *node similarity* as *(structural) equivalence*

$$\sigma_{ij} \sim |\Gamma_i \cap \Gamma_j|$$

1. *blockmodeling* by *(hierarchical) clustering* $\mathcal{O}(n^2)$
2. return *block model* at desired *clustering resolution*



javax adjacency matrix



javax block model

‡ `javax.swing`, `javax.management`, `javax.naming`, `javax.print`, `javax.xml`, `javax.lang` etc.

# blockmodeling *structural*

*similar* nodes have *same neighbors*

— *standard structural equivalence* [LW71] of $i$ and $j$ is

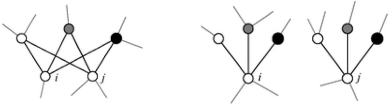
$$\sigma_{ij} = \sum_x A_{ix} A_{xj} = |\Gamma_i \cap \Gamma_j|$$

— *Salton structural equivalence* [SM83] of $i$ and $j$ is

- $\theta_{ij}$ is *angle* between neighborhoods $A_i$ and $A_j$

$$\sigma_{ij} = \cos\theta_{ij} = \frac{\sum_x A_{ix} A_{xj}}{\sqrt{\sum_x A_{ix}^2}\sqrt{\sum_x A_{jx}^2}} = \frac{|\Gamma_i \cap \Gamma_j|}{\sqrt{k_i k_j}}$$

— *Leicht structural equivalence* [LHN06] of $i$ and $j$ is $\sigma_{ij} = \frac{|\Gamma_i \cap \Gamma_j|}{k_i k_j / n}$




structural regular equivalence

# blockmodeling *regular*

*similar* nodes have *equivalent* *neighbors*

— *standard regular equivalence* [WR83] of $i$ and $j$ is

– $\alpha < \lambda^{-1}$ is *positive constant* and $\lambda$ *leading eigenvalue* of $A$

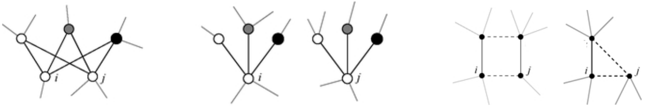
$$\sigma_{ij} = \alpha \sum_{xy} A_{ix} A_{jy} \sigma_{xy} + \delta_{ij} = \alpha \sum_{x \in \Gamma_i} \sum_{y \in \Gamma_j} \sigma_{xy} + \delta_{ij}$$

$\sigma = \alpha A \sigma A + I$ and thus $\sigma^{(0)} = 0$, $\sigma^{(1)} = I$, $\sigma^{(2)} = \alpha A^2 + I$, $\sigma^{(3)} = \alpha^2 A^4 + \alpha A^2 + I$ etc.

— *Katz regular equivalence* [Kat53] of $i$ and $j$ is

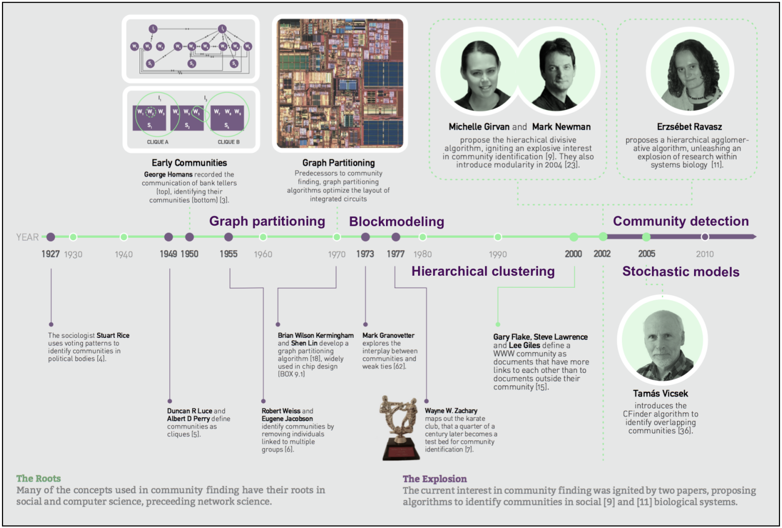
$$\sigma_{ij} = \alpha \sum_{x} A_{ix} \sigma_{xj} + \delta_{ij} = \alpha \sum_{x \in \Gamma_i} \sigma_{xj} + \delta_{ij}$$

$\sigma = \alpha A \sigma + I$ and thus $\sigma^{(0)} = 0$, $\sigma^{(1)} = I$, $\sigma^{(2)} = \alpha A + I$, $\sigma^{(3)} = \alpha^2 A^2 + \alpha A + I$ etc.




structural regular equivalence standard Katz

# blockmodeling *history*



# *stochastic* models

introduction to *network analysis* (*ina*)

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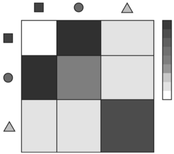
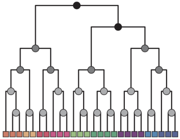
# stochastic *models*

- *random graph model* $G(n, m)$ for *network links* $m$ [ER59]
- *configuration model* $G(\{k\})$ for *node degrees* $\{k\}$ [NSW01]
- *exponential* $p^*$*-model* $G(n, \{\langle x \rangle\})$ for *any expectations* $\{\langle x \rangle\}$

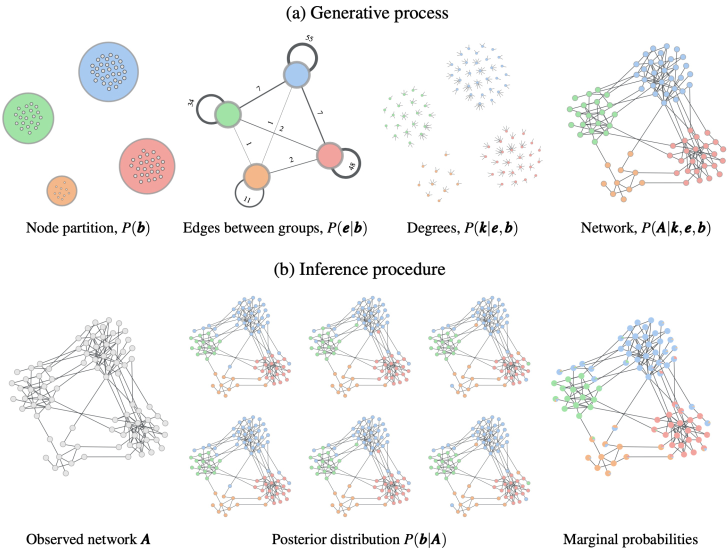
- *stochastic block model* $G(\{C\})$ for *node clusters* $\{C\}$ [HLL83]
- *hierarchical model* $G(H)$ for *node hierarchy* $H$ [CMN08]





*assortative & disassortative stochastic block models

# stochastic *process*



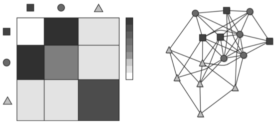
# stochastic $G(\{C\})$ model

- $G(\{C\},\{p\})$ *stochastic block model* [HLL83]
- *link* between $i$ and $j$ *placed with probability* $p_{c_i c_j}$
  - $m_{c_i c_j}$ is *number of links* between $C_i$ and $C_j$
  - $M_{c_i c_j}$ is *maximum* $m_{c_i c_j}$ hence $n_i n_j$ or $\binom{n_i}{2}$

$$\mathrm{P}(A|\{C\},\{p\}) = \prod_{i\leq j} p_{c_i c_j}^{A_{ij}}(1-p_{c_i c_j})^{1-A_{ij}} = \prod_{c_i\leq c_j} p_{c_i c_j}^{m_{c_i c_j}}(1-p_{c_i c_j})^{M_{c_i c_j}-m_{c_i c_j}}$$

- *maximum likelihood* $G(\{C\})$ *block model*
  - $\frac{m_{c_i c_j}}{M_{c_i c_j}}$ is *maximum likelihood estimate* for $p_{c_i c_j}$

$$\mathcal{L}(A|\{C\}) = \log \mathrm{P}(A|\{C\}) = \sum_{c_i\leq c_j} m_{c_i c_j} \log \frac{m_{c_i c_j}}{M_{c_i c_j}-m_{c_i c_j}} + M_{c_i c_j} \log \frac{M_{c_i c_j}-m_{c_i c_j}}{M_{c_i c_j}}$$



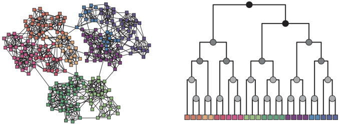
see graph-tool implementation

# stochastic $G(H)$ *model*

- $G(H,\{p\})$ *hierarchical model* [CMN08]
- *link* between $i$ and $j$ *placed with probability* $p_{r_{ij}}$
  - $r$ is *root with subtrees* $L_r, R_r$ and $r_{ij}$ *lowest root* of $i$ and $j$
  - $m_r$ is *number of links* between $L_r, R_r$ and $M_r$ is $|L_r||R_r|$

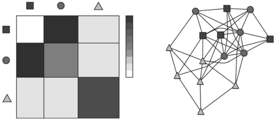
    $\mathrm{P}(A|H,\{p\}) = \prod_{i\leq j} p_{r_{ij}}^{A_{ij}}(1-p_{r_{ij}})^{1-A_{ij}} = \prod_r p_r^{m_r}(1-p_r)^{M_r-m_r}$
- *maximum likelihood* $G(H)$ *hierarchical model*
  - $\frac{m_r}{M_r}$ is *maximum likelihood estimate* for $p_r$

    $\mathcal{L}(A|H) = \log \mathrm{P}(G|H) = \sum_r m_r \log \frac{m_r}{M_r-m_r} + M_r \log \frac{M_r-m_r}{M_r}$

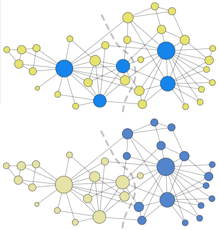


see **randomgraphs** implementation

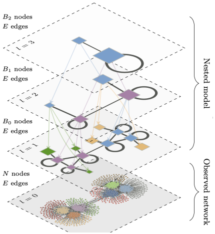
# stochastic *overview*



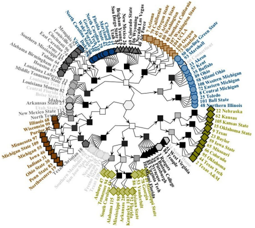
*stochastic block model* [HLL83]



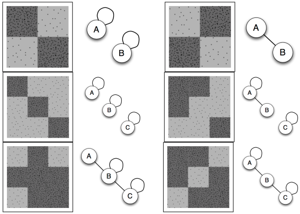
*degree-corrected SBM* [KN11]




*nested SBM* [Pei15]



*hierarchical models* [CMN08, ŠB14]



*role models* [RW07, NL07, GSPA07]

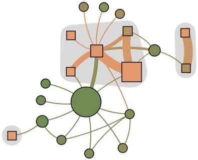


*image graphs* [ŠB12]

† overlapping & corrected models also known as mixture & mixed membership models

# blockmodeling *references*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