network *blockmodeling*

introduction to *network analysis* (*ina*)

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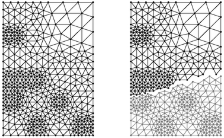
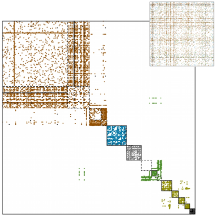
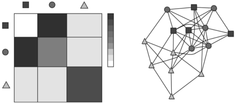
blockmodeling *overview*



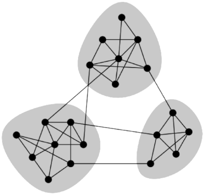
graph partitioning [KL70, Fie73]



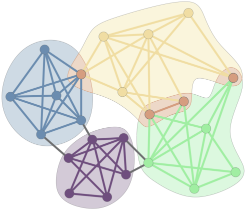
blockmodeling [LW71, WR83]



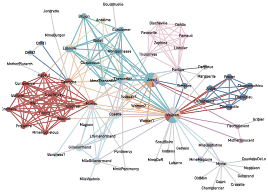
stochastic block model [Pei15]



communities [GN02]



overlapping communities [PDFV05]



link communities [EL09, ABL10]

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* assortative & disassortative equivalence blockmodeling

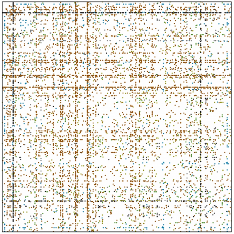
blockmodeling *equivalence*

— *standard equivalence blockmodeling* [DBF05]

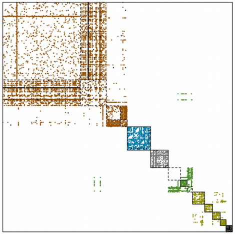
– define *node similarity* as *(structural) equivalence*

$$\sigma_{ij} \sim |\Gamma_i \cap \Gamma_j|$$

1. *blockmodeling* by *(hierarchical) clustering* $\mathcal{O}(n^2)$
2. return *block model* at desired *clustering resolution*



javax adjacency matrix



javax block model

‡ javax.swing, javax.management, javax.naming, javax.print, javax.xml, javax.lang etc.

blockmodeling *structural*

similar nodes have *same neighbors*

— *standard structural equivalence* [LW71] of i and j is

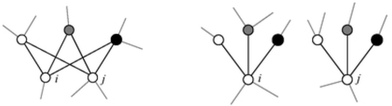
$$\sigma_{ij} = \sum_x A_{ix} A_{xj} = |\Gamma_i \cap \Gamma_j|$$

— *Salton structural equivalence* [SM83] of i and j is

  – θ_{ij} is *angle* between neighborhoods A_i and A_j

$$\sigma_{ij} = \cos\theta_{ij} = \frac{\sum_x A_{ix} A_{xj}}{\sqrt{\sum_x A_{ix}^2}\sqrt{\sum_x A_{jx}^2}} = \frac{|\Gamma_i \cap \Gamma_j|}{\sqrt{k_i k_j}}$$

— *Leicht structural equivalence* [LHN06] of i and j is $\sigma_{ij} = \frac{|\Gamma_i \cap \Gamma_j|}{k_i k_j / n}$



structural　　　regular equivalence

blockmodeling *regular*

similar nodes have *equivalent* *neighbors*

— *standard regular equivalence* [WR83] of i and j is

- $\alpha < \lambda^{-1}$ is *positive constant* and λ *leading eigenvalue* of A

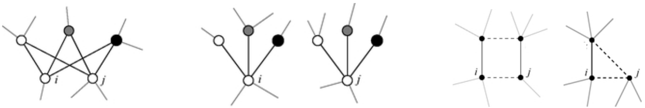
$$\sigma_{ij} = \alpha \sum_{xy} A_{ix} A_{jy} \sigma_{xy} + \delta_{ij} = \alpha \sum_{x \in \Gamma_i} \sum_{y \in \Gamma_j} \sigma_{xy} + \delta_{ij}$$

$\sigma = \alpha A \sigma A + I$ and thus $\sigma^{(0)} = 0$, $\sigma^{(1)} = I$, $\sigma^{(2)} = \alpha A^2 + I$, $\sigma^{(3)} = \alpha^2 A^4 + \alpha A^2 + I$ etc.

— *Katz regular equivalence* [Kat53] of i and j is

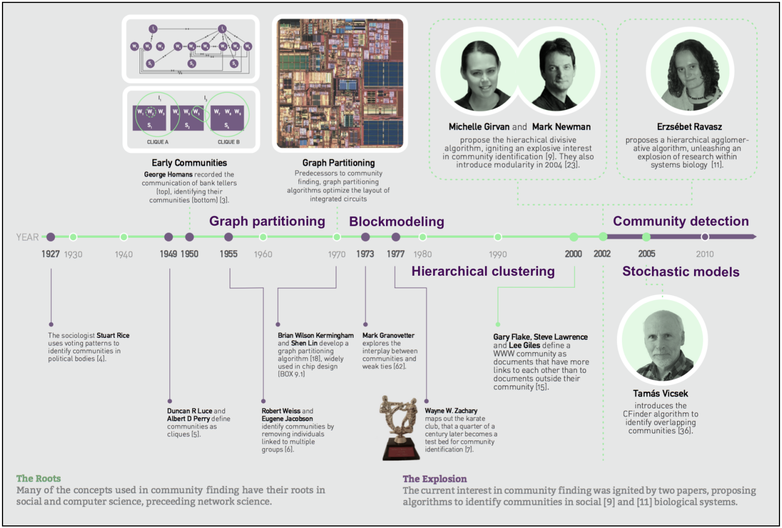
$$\sigma_{ij} = \alpha \sum_{x} A_{ix} \sigma_{xj} + \delta_{ij} = \alpha \sum_{x \in \Gamma_i} \sigma_{xj} + \delta_{ij}$$

$\sigma = \alpha A \sigma + I$ and thus $\sigma^{(0)} = 0$, $\sigma^{(1)} = I$, $\sigma^{(2)} = \alpha A + I$, $\sigma^{(3)} = \alpha^2 A^2 + \alpha A + I$ etc.



structural regular equivalence standard Katz

blockmodeling *history*



stochastic models

introduction to *network analysis* (*ina*)

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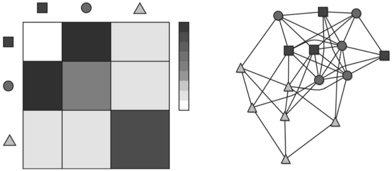
stochastic *models*

— *random graph model* $G(n, m)$ for *network links* m [ER59]

— *configuration model* $G(\{k\})$ for *node degrees* $\{k\}$ [NSW01]

— *exponential* p^**-model* $G(n, \{\langle x \rangle\})$ for *any expectations* $\{\langle x \rangle\}$

— *stochastic block model* $G(\{C\})$ for *node clusters* $\{C\}$ [HLL83]



*assortative & disassortative stochastic block models

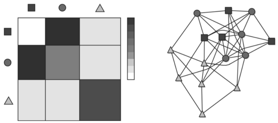
stochastic $G(\{C\})$ *model*

- $G(\{C\},\{p\})$ *stochastic block model* [HLL83]
- *link* between i and j *placed with probability* $p_{c_i c_j}$
  - $m_{c_i c_j}$ is *number of links* between C_i and C_j
  - $M_{c_i c_j}$ is *maximum* $m_{c_i c_j}$ hence $n_i n_j$ or $\binom{n_i}{2}$

$$\mathrm{P}(A|\{C\},\{p\}) = \prod_{i\leq j} p_{c_i c_j}^{A_{ij}} (1-p_{c_i c_j})^{1-A_{ij}} = \prod_{c_i \leq c_j} p_{c_i c_j}^{m_{c_i c_j}} (1-p_{c_i c_j})^{M_{c_i c_j}-m_{c_i c_j}}$$

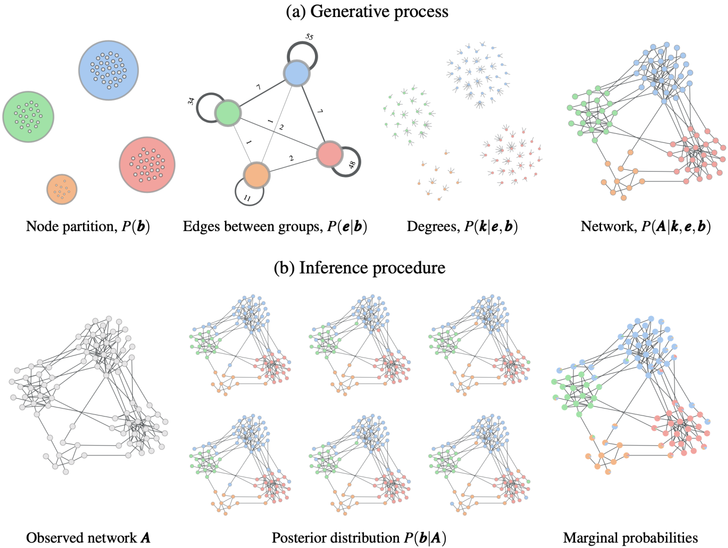
- *maximum likelihood* $G(\{C\})$ *block model*
  - $\frac{m_{c_i c_j}}{M_{c_i c_j}}$ is *maximum likelihood estimate* for $p_{c_i c_j}$

$$\mathcal{L}(A|\{C\}) = \log \mathrm{P}(A|\{C\}) = \sum_{c_i \leq c_j} m_{c_i c_j} \log \frac{m_{c_i c_j}}{M_{c_i c_j}-m_{c_i c_j}} + M_{c_i c_j} \log \frac{M_{c_i c_j}-m_{c_i c_j}}{M_{c_i c_j}}$$

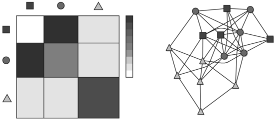


see graph-tool implementation

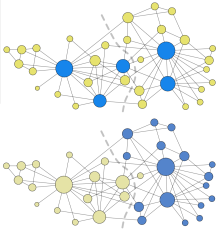
stochastic *process*



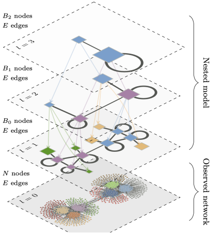
stochastic *overview*



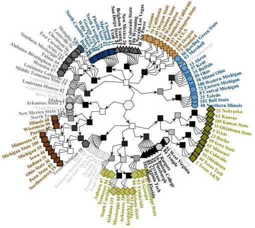
stochastic block model [HLL83]



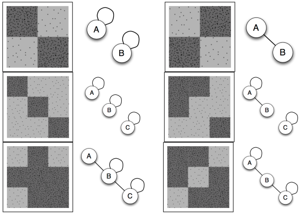
degree-corrected SBM [KN11]



nested SBM [Pei15]



hierarchical models [CMN08, ŠB14]



role models [RW07, NL07, GSPA07]

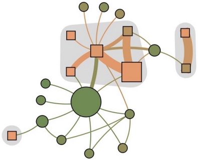


image graphs [ŠB12]

† overlapping & corrected models also known as mixture & mixed membership models

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