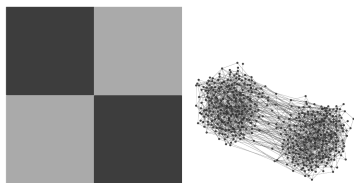


*core-periphery* structure

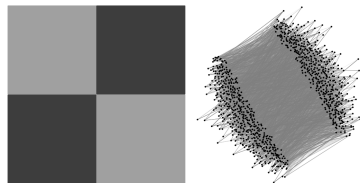
introduction to *network analysis* (*ina*)

Lovro Šubelj  
University of Ljubljana  
spring 2023/24

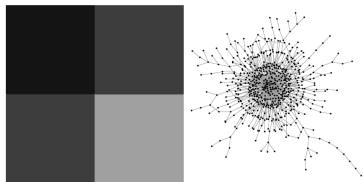
# core-periphery *block model*



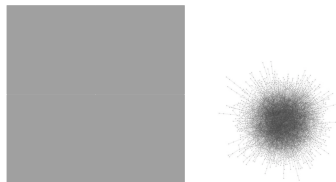
*community block model* [GN02]



*disassortative (bipartite) block model* [NL07]



*core-periphery block model* [Sei83]



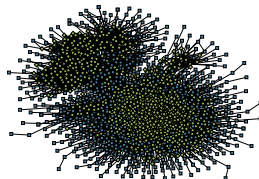
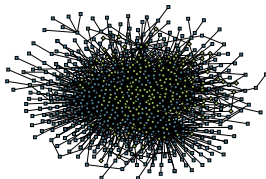
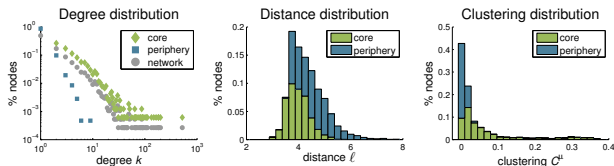
*random block model* [ER59]

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\* origin of core-periphery structure in international relations

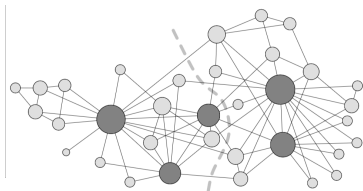
# core-periphery *structure*

- *core/periphery nodes* have *higher/lower degrees*  $k$
- *core/periphery nodes* are on *shorter/longer distances*  $\ell$
- *core/periphery nodes* have *higher/lower clustering*  $C^i$

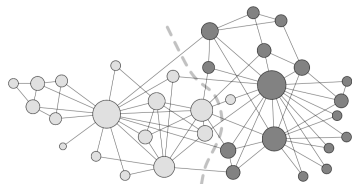


## core-periphery *stochastic*

- $G(\{C_1, C_2\}, \{p_{11}, p_{12}, p_{22}\})$  *stochastic block model* [HLL83]
  - $n_i$  is *size* of *cluster*  $C_i$  &  $p_{ij}$  is *link density* between  $C_i$  and  $C_j$
- *density-based core-periphery* structure when  $p_{11} \gg p_{12} \gg p_{22}$
- *lookalike core-periph.* when  $n_1 p_{11} \gg 1$ ,  $n_1 p_{12} \ll 1$ ,  $n_2 p_{22} \approx 1$



*non-corrected block model*  $p_{11} > p_{12} > p_{22}$



*degree-corrected block model*  $p_{11} \approx p_{22} > p_{12}$

## core-periphery *discrete/continuous*

— *discrete core-periphery division*  $\delta \in \{0, 1\}$  [BE00]

–  $\delta_i = 1$  for *core nodes*  $i$  &  $\delta_i = 0$  for *peripheral nodes*  $i$

$$\rho_{\{0,1\}} = \sum_{ij} A_{ij} \Delta_{ij} \quad \Delta_{ij} = \begin{cases} 1 & \text{if } \delta_i = \delta_j = 1 \\ 0 & \text{if } \delta_i = \delta_j = 0 \\ \in [0, 1] & \text{if } \delta_i - \delta_j \neq 0 \end{cases}$$

— *continuous core-periphery centrality*  $\delta \in [0, 1]$

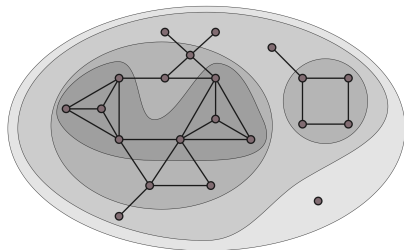
–  $\delta_i \approx 1$  for *core nodes*  $i$  &  $\delta_i \approx 0$  for *peripheral nodes*  $i$

$$\rho_{[0,1]} = \sum_{ij} A_{ij} \delta_i \delta_j$$

$$\Delta^1 = \left[ \begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{array} \right] \quad \Delta^\alpha = \left[ \begin{array}{ccc|ccc} 0 & 1 & 1 & \alpha & \alpha & \alpha \\ 1 & 0 & 1 & \alpha & \alpha & \alpha \\ 1 & 1 & 0 & \alpha & \alpha & \alpha \\ \hline \alpha & \alpha & \alpha & 0 & 0 & 0 \\ \alpha & \alpha & \alpha & 0 & 0 & 0 \\ \alpha & \alpha & \alpha & 0 & 0 & 0 \end{array} \right] \quad \delta = \left[ \begin{array}{c} 1 \\ 0.8 \\ 0.7 \\ \hline 0.4 \\ 0.2 \\ 0.1 \end{array} \right]$$

## core-periphery *k*-cores

- *k*-cores are *subgraphs of nodes* with  $\geq k$  neighbors [Sei83]  
remove nodes with degree  $< k$  until no such node remains [BZ11]
- *k*-shells are *nodes of k-cores* that are *not in k + 1-cores*
- *k*-cores are *nested* while *k*-shells form *decomposition*



1-cores are connected components w/o isolates & *k*-cores can be disconnected

## core-periphery $k^*$ -core

- Holme's  $k^*$ -core maximizes closeness centrality  $\ell^{-1}$  [Hol05]
  - $d_{ij}$  is distance between  $i$  and  $j$  &  $\ell_i$  is farness centrality of  $i$
  - $\ell_C^{-1}$  is closeness centrality of cluster  $C$  &  $n_c$  is size of  $C$

$$\ell_i = \frac{1}{n-1} \sum_{j \neq i} d_{ij} \qquad \ell_C^{-1} = \left( \frac{1}{n_c} \sum_{i \in C} \ell_i \right)^{-1}$$

- Holme's core-periphery coefficient  $c_{cp}$  for  $k^*$ -core
  - $N$  is set of nodes &  $N_k$  are nodes in  $k$ -core
  - $\langle \dots \rangle_{G'}$  is expectation in random graph  $G'$

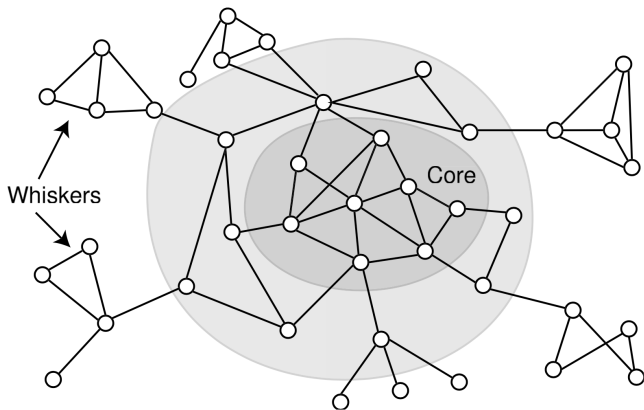
$$c_{cp} = \ell_{N_{k^*}}^{-1} / \ell_N^{-1} - \left\langle \ell_{N'_{k^*}}^{-1} / \ell_{N'}^{-1} \right\rangle_{G'}$$

# core-periphery *coefficient*

Network	$N$	$M$	$c_{cp}$	
Geographical networks	Interstate highways	935	1315	0.231(1)
	Pipelines	2999	3079	0.180(2)
	Streets, Stockholm	3325	5100	0.255(1)
	Streets, Göteborg	1258	1516	0.040(3)
	Airport	449	2795	0.0523(3)
	Internet	1968(66)	4051(121)	0.045(2)
One-mode projections of affiliation networks	arXiv	48561	287570	-0.08(3)
	Board of directors	6193	43074	-0.037(2)
	Ajou University students	7285(128)	75898(6566)	-0.08(1)
Acquaintance networks	High School friendship	571(43)	1078(85)	0.006(7)
	Prisoners	58	83	-0.043(2)
	Social scientists	34	265(35)	-0.002(4)
Electronic communication	e-mail, Ebel <i>et al.</i>	39592	57703	-0.229(4)
	e-mail, Eckmann <i>et al.</i>	3186	31856	-0.091(2)
	Internet community, nioki.com	49801	239265	-0.014(2)
	Internet community, pussokram.com	28295	115335	-0.183(5)
Reference networks	WWW, nd.edu	325729	1090108	-0.027(3)
	HEP citations	27400	352021	-0.10(1)
Software dependencies	GNU / Linux	504	793	-0.155(1)
Food webs	Little Rock Lake	92	960	0.005(6)
	Ythan Estuary	134	593	-0.020(1)
Neural network	<i>C. elegans</i>	280	1973	0.040(6)
Biochemical networks	<i>Drosophila</i> protein	2915	4121	-0.035(2)
	<i>S. cerevisiae</i> protein	3898	7283	-0.249(1)
	<i>S. cerevisiae</i> genetic	1503	5043	-0.0646(7)
	Metabolic networks	427(27)	1257(88)	-0.002(6)
	Whole cellular networks	623(32)	1752(103)	-0.004(6)



## core-periphery *nestedness*



*nested cores & whiskers communities* [LLDM09, YL13]

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