subgraphs or *fragments*

introduction to *network analysis* (*ina*)

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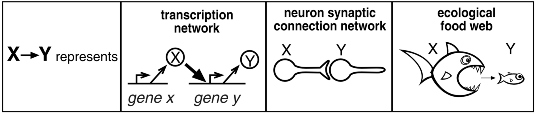
fragments *definition*

— small *subgraphs* are *building blocks* of networks

— *subgraphs* characterize *local network structure*



— *fragments* = *connected subgraphs* of networks [EK15]

— *motifs* = *frequent non-induced* subgraphs [MSOI+02]

— *graphlets* = *specific induced* subgraphs [PCJ04]

see mfinder and orca for implementations

network *motifs*

introduction to *network analysis* (*ina*)

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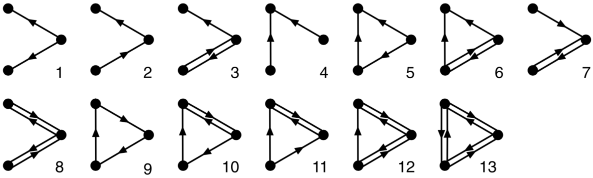
motifs *definition*

- *fragments* characterize *network-wise local structure*
- *motifs* are *frequent non-induced fragments* [MSOI+02]

  probability of *motif appearing in random graph*

  equal or greater number of times is < 0.01
- (*un*)*directed motifs* consisting of *three to five/six/seven nodes*



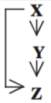
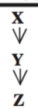
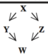
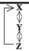
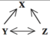
all 13 directed three-node motifs

motifs *significance*

- *motif significance* Z with *normal distribution* $N(0,1)$
  - $\widetilde{n}_i$ is *number of motifs* i in *random graph* with *variance* $\widetilde{\sigma}_i^2$
  - n_i is *number of motifs* i in *real network*

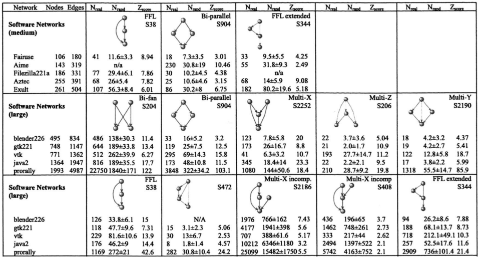
$$Z_i = \frac{n_i - \langle \widetilde{n}_i \rangle}{\widetilde{\sigma}_i} \qquad\qquad n_i - \langle \widetilde{n}_i \rangle > 0.1 \langle \widetilde{n}_i \rangle$$

- $\widetilde{n}/\widetilde{\sigma}$ estimated by *motif preserving randomization* [MSOI+02]

Network	Nodes	Edges	N_{real}	$N_{rand} \pm SD$	Z score	N_{real}	$N_{rand} \pm SD$	Z score	N_{real}	$N_{rand} \pm SD$	Z score
Neurons					**Feed-forward loop**			**Bi-fan**			**Bi-parallel**
C. elegans†	252	509	125	90 ± 10	3.7	127	55 ± 13	5.3	227	35 ± 10	20
Food webs					**Three chain**			**Bi-parallel**			
Little Rock	92	984	3219	3120 ± 50	2.1	7295	2220 ± 210	25			
Ythan	83	391	1182	1020 ± 20	7.2	1357	230 ± 50	23			
St. Martin	42	205	469	450 ± 10	NS	382	130 ± 20	12			
Electronic circuits (forward logic chips)					**Feed-forward loop**			**Bi-fan**			**Bi-parallel**
s15850	10,383	14,240	424	2 ± 2	285	1040	1 ± 1	1200	480	2 ± 1	335
s38584	20,717	34,204	413	10 ± 3	120	1739	6 ± 2	800	711	9 ± 2	320
s38417	23,843	33,661	612	3 ± 2	400	2404	1 ± 1	2550	531	2 ± 2	340
World Wide Web					**Feedback with two mutual dyads**			**Fully connected triad**			**Uplinked mutual dyad**
nd.edu§	325,729	1.46e6	1.1e5	2e3 ± 1e2	800	6.8e6	5e4±4e2	15,000	1.2e6	1e4 ± 2e2	5000

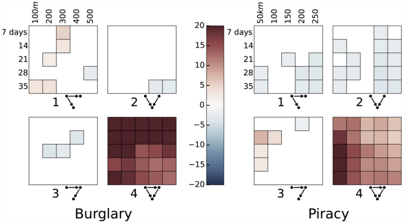
motifs *examples*

motif Z-scores of class *software networks* [VS05]



Network	Nodes	Edges	N_{real}	N_{rand}	Z_{score}	N_{real}	N_{rand}	Z_{score}	N_{real}	N_{rand}	Z_{score}	N_{real}	N_{rand}	Z_{score}	N_{real}	N_{rand}	Z_{score}
Software Networks (medium)					FFL S38			Bi-parallel S904			FFL extended S344						
Fairuse	106	180	41	11.6±3.3	8.94	18	7.3±3.5	3.01	33	9.5±5.5	4.25						
Aime	143	319		n/a		230	30.8±19	10.46	55	31.8±9.3	2.49						
Filezilla221a	186	331	77	29.4±6.1	7.86	30	10.2±4.5	4.38		n/a							
Aztec	255	391	68	26±5.4	7.82	25	10.6±4.6	3.15	68	14±5.9	9.08						
Exult	261	504	107	56.3±8.4	6.01	86	30.2±8	6.75	182	80.2±19.6	5.18						
Software Networks (large)					Bi-fan S204			Bi-parallel S904			Multi-X S2252			Multi-Z S206			Multi-Y S2190
blender226	495	834	486	138±30.3	11.4	33	16±5.2	3.2	123	7.8±5.8	20	22	3.7±3.6	5.04	18	4.2±3.2	4.37
gtk221	748	1147	644	189±33.8	13.4	119	25±7.5	12.5	173	26±16.7	8.8	21	2.0±1.7	10.9	19	4.2±2.7	5.41
vtk	771	1362	512	262±39.9	6.27	295	69±14.3	15.8	41	6.3±3.2	10.7	193	27.7±14.7	11.2	122	12.8±5.8	18.7
java2	1364	1947	816	189±35.5	17.7	173	48±10.8	11.5	345	18.4±14	23.3	22	2.2±2.1	9.5	17	3.8±2.2	5.99
prorally	1993	4987	22750	1840±171	122	3848	322±34.2	103.1	1080	144±50.6	18.4	210	28.7±9.2	19.8	1318	55.5±14.7	85.9
Software Networks (large)					FFL S38			S472			Multi-X incomp. S2186			Multi-X incomp S408			FFL extended S344
blender226			126	33.8±6.1	15		N/A		1976	766±162	7.43	436	196±65	3.7	94	26.2±8.6	7.88
gtk221			118	47.7±9.6	7.31	15	3.1±2.3	5.06	4177	1941±398	5.6	1462	748±261	2.73	188	68.1±13.7	8.73
vtk			229	81.6±10.6	13.9	30	13±6.7	2.53	707	388±61.6	5.17	333	217±44	2.62	718	212.1±49.1	10.3
java2			176	46.2±9	14.4	8	1.8±1.4	4.57	10212	6346±1180	3.2	2494	1397±522	2.1	257	52.5±17.6	11.6
prorally			1169	272±21	42.6	282	30.8±10.4	24.2	25099	15482±1750	5.5	5742	4163±752	2.1	2909	736±101.4	21.4

motif Z-scores of spatio-temporal *crime networks* [DM15]



motifs *profiles*

- *motif significance profile SP* [MSOI+02] defined as
  - Z_i is *significance of motif i* in *real network*

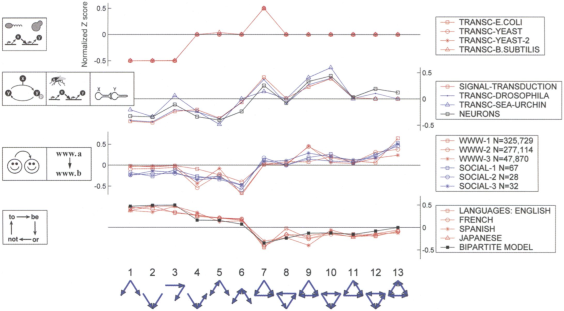
$$SP_i = \frac{Z_i}{\sqrt{\sum_i Z_i^2}} \qquad Z_i = \frac{n_i - \langle \widetilde{n}_i \rangle}{\widetilde{\sigma}_i} \qquad n_i \geq 4$$

- *motif abundance/ratio profile RP* [MIK+04] defined as
  - A_i is *abundance of motif i* in *real network*

$$RP_i = \frac{A_i}{\sqrt{\sum_i A_i^2}} \qquad A_i = \frac{n_i - \langle \widetilde{n}_i \rangle}{n_i + \langle \widetilde{n}_i \rangle + \epsilon_i} \qquad \epsilon_i = 4$$

motifs *families*

- directed *motif significance profiles* [MSOI+02]
- profiles reveal (*super*)*families of real networks*



all 13 directed three-node motifs

network *graphlets*

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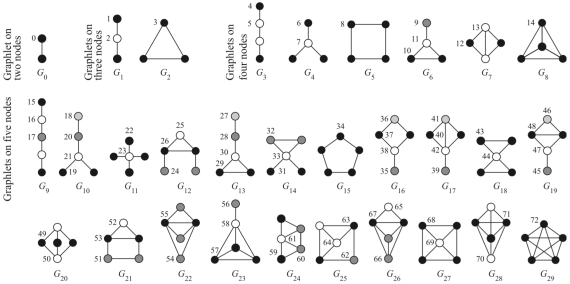
graphlets *definition*

- *fragments* characterize *node-wise local structure*
- *graphlets* are *specific induced fragments* [PCJ04]
- *graphlet orbits* are *automorphisms of graphlets* [Prž07]
- (*un*)*directed graphlets* consisting of *three to five/... nodes*

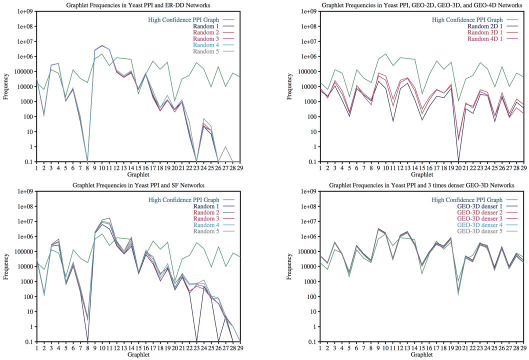


all 30 undirected two- to five-node graphlets with 73 orbits

graphlets *frequency*

- *relative graphlet frequency* F [PCJ04] defined as
  - n_i is *number of graphlets* i in *real network*

$$F_i = \frac{n_i}{\sum_i n_i}$$



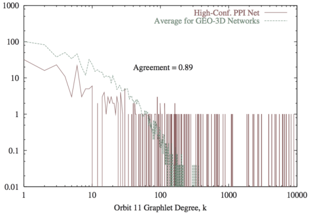
graphlet frequency in protein network and random graphs

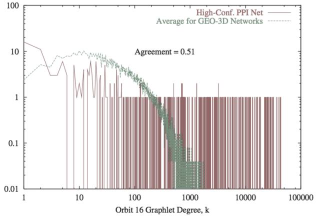
graphlets *distribution*

— *i-th orbit graphlet degree distribution* p_k^i [Prž07] defined as

- p_k^0 is *degree distribution* p_k of *real network*
- p_k^i is *graphlet degree distribution* for *i-th orbit*
- $\widetilde{p}_k^i$ is *scaled graphlet degree distribution* for *i-th orbit*

$$\widetilde{p}_k^i \sim p_k^i / k \qquad \widetilde{p}_k^0 = p_k^0 = p_k$$





11th and 16th orbit graphlet degree distributions of protein network and random graph

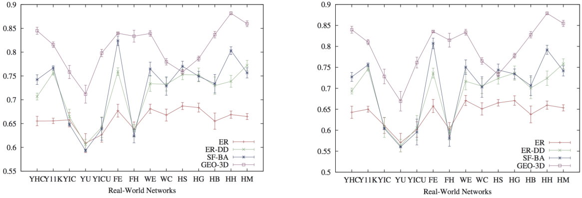
graphlets *agreement*

- *i-th orbit graphlet agreement* A_i [Prž07] defined as
  - $\widetilde{p}_k^i$ is *i-th orbit graphlet degree distribution* of *first network*
  - $\widetilde{q}_k^i$ is *i-th orbit graphlet degree distribution* of *second network*

$$A_i = 1 - \sqrt{\frac{1}{2}\sum_k \left(\log \widetilde{q}_k^i - \log \widetilde{p}_k^i\right)^2}$$

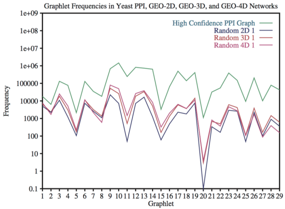
- *arithmetic/geometric graphlet agreement* A defined as

$$A = \frac{1}{73}\sum_i A_i \qquad\qquad A = \left(\prod_i A_i\right)^{\frac{1}{73}}$$

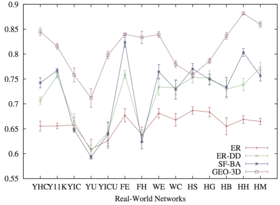


arithmetic/geometric graphlet agreement of protein networks and random graphs

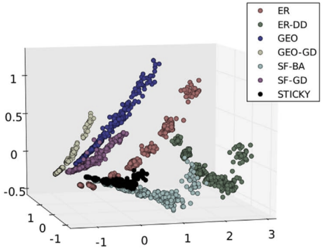
graphlets *measures*

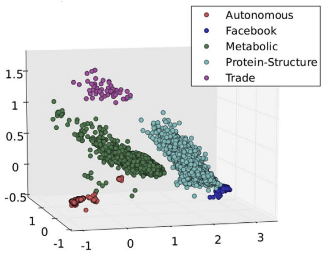


relative graphlet frequency [PCJ04]



graphlet distribution agreement [Prž07]





graphlet correlation matrix and *distance* [YMDD^{+}14]

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