scale-free networks

introduction to *network analysis* (*ina*)

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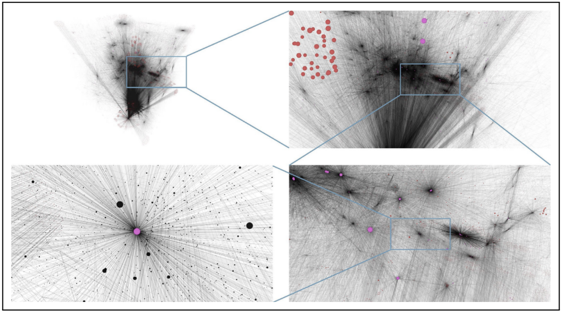
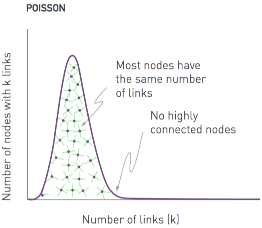
scale-free *property*

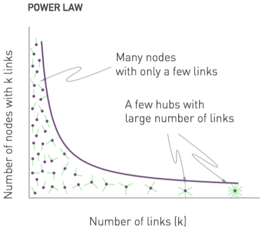
- — *random graphs* = *Poisson degree distribution* p_k [ER59]
- — *real networks* contain *highly linked hubs* [Pri65, FFF99]
- — *scale-free networks* $\sim$ *power-law degree distribution* p_k [BA99]

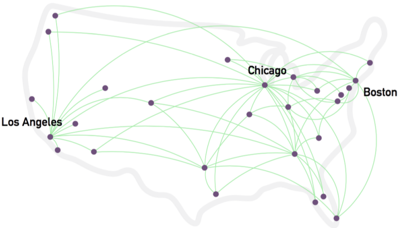


see zooming into World Wide Web demo

scale-free *structure*





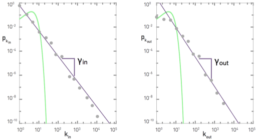


scale-free *power-law*

— *power-law degree distribution* p_k with *exponent* $\gamma > 1$

$$p_k \sim k^{-\gamma}$$

$$\log p_k \sim -\gamma \log k$$



— *theoretically correct discrete power-law* p_k for $k \geq 1$

$$\sum_{k=1}^{\infty} p_k = C \sum_{k=1}^{\infty} k^{-\gamma} = C\zeta(\gamma) = 1$$

$$p_k = Ck^{-\gamma} = \frac{k^{-\gamma}}{\zeta(\gamma)}$$

— *analytically convenient continuos power-law* $p(k)$ for $k \geq k_{min}$

$$\int_{k_{min}}^{\infty} p(k)\, \mathrm{d}k = C \int_{k_{min}}^{\infty} k^{-\gamma}\, \mathrm{d}k = C \left. \frac{k^{-\gamma+1}}{-\gamma+1} \right|_{k_{min}}^{\infty} = C \frac{k_{min}^{-\gamma+1}}{\gamma-1} = 1$$

$$p(k) = Ck^{-\gamma} = (\gamma - 1)k_{min}^{\gamma-1} k^{-\gamma}$$

scale-free *hubs*

— for *small* $k \ll \langle k \rangle$ *power-law above Poisson*

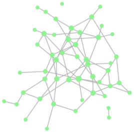
many *small degree nodes* in *scale-free network*

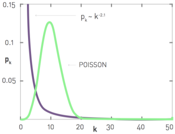
— for *average* $k \approx \langle k \rangle$ *power-law below Poisson*

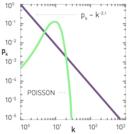
most *nodes similar degree* in *random graph*

— for *large* $k \gg \langle k \rangle$ *power-law above Poisson*

existence of hubs in *scale-free network*







— *random graph* with $n \approx 10^{12}$ and $\langle k \rangle = 4.6$ then $n_{k \geq 100} \approx 10^{-82}$

— *scale-free network* with $n \approx 10^{12}$ and $\gamma = 2.1$ then $n_{k \geq 100} \approx 4 \cdot 10^9$

scale-free *cutoff*

- *maximum degree* k_{max} by *upper natural cutoff* of $p(k)$
- for *random graph* with *exponential* $p(k) = \lambda e^{\lambda k_{min}} e^{-\lambda k}$

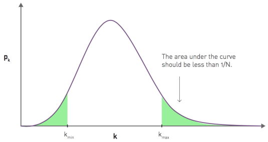
$$\int_{k_{max}}^{\infty} p(k)\,\mathrm{d}k = \lambda e^{\lambda k_{min}} \left.\frac{e^{-\lambda k}}{-\lambda}\right|_{k_{max}}^{\infty} = e^{\lambda k_{min}} e^{-\lambda k_{max}} = n^{-1}$$

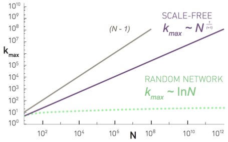
$$k_{max} = k_{min} + \frac{\ln n}{\lambda}$$

- for *scale-free network* with *power-law* $p(k) = (\gamma - 1) k_{min}^{\gamma-1} k^{-\gamma}$

$$\int_{k_{max}}^{\infty} p(k)\,\mathrm{d}k = (\gamma - 1) k_{min}^{\gamma-1} \left.\frac{k^{-\gamma+1}}{-\gamma+1}\right|_{k_{max}}^{\infty} = k_{min}^{\gamma-1} k_{max}^{-\gamma+1} = n^{-1}$$

$$k_{max} = k_{min} n^{\frac{1}{\gamma-1}}$$





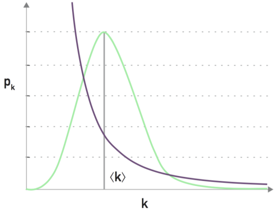
- *random graph* with $n \approx 3 \cdot 10^5$ and $\lambda = 1$ then $k_{max} \approx 14$
- *scale-free network* with $n \approx 3 \cdot 10^5$ and $\gamma = 2.1$ then $k_{max} \approx 10^5$

scale-free *moments*

- *x-th moment* $\langle k^x \rangle$ of *power-law* $p_k \sim k^{-\gamma}$
  - $\langle k^2 \rangle = \sigma_k^2 + \langle k \rangle^2$ determines *spread* and $\langle k^3 \rangle$ determines *skewness*

$$\langle k^x \rangle = \sum_{k=1}^{\infty} k^x p_k \approx \int_{k_{min}}^{k_{max}} k^x p(k)\,\mathrm{d}k \sim \frac{k_{max}^{x-\gamma+1} - k_{min}^{x-\gamma+1}}{x-\gamma+1}$$

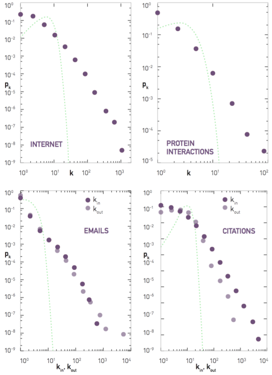
- *moments* $x \leq \gamma - 1$ *finite* whereas *moments* $x > \gamma - 1$ *diverge*

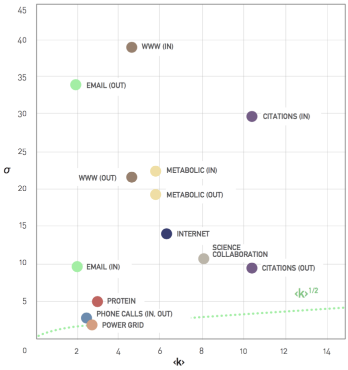


- *scale-free networks* $\gamma < 3$ *lack scale* as $k = \langle k \rangle \pm \infty$
- *random graphs have scale* as $k = \langle k \rangle \pm \sqrt{\langle k \rangle}$

scale-free *networks*

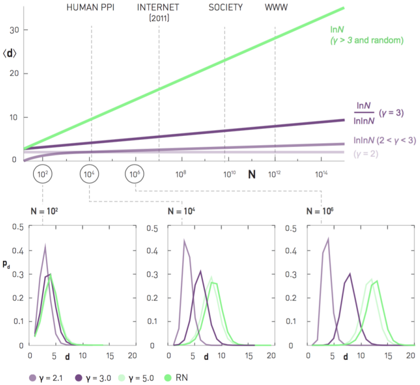
- *heavy-tail* p_k of real networks [Bar16]
- *spread* $\sigma_k = \sqrt{\langle k^2 \rangle - \langle k \rangle^2}$ in real networks



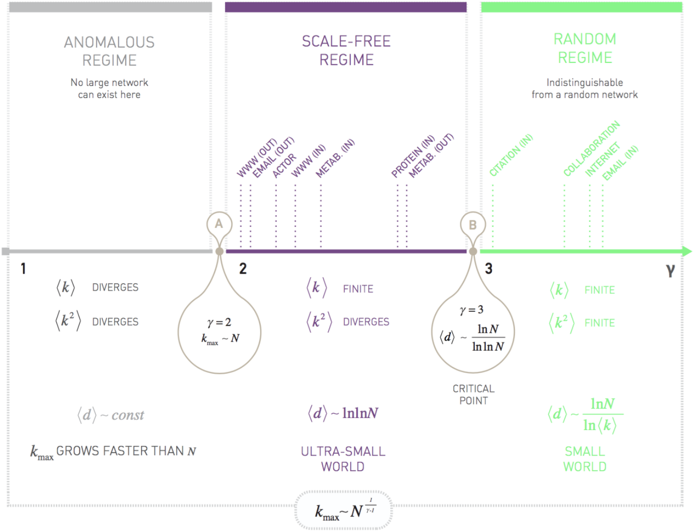


scale-free *"small-world"*

- *random graphs* are *"small-world"* as $\langle d \rangle \simeq \frac{\ln n}{\ln \langle k \rangle}$
- *scale-free networks* $\gamma > 3$ are *"small-world"* as $\langle d \rangle \sim \ln n$
- *scale-free networks* $\gamma < 3$ *"ultrasmall-world"* as $\langle d \rangle \sim \ln \ln n$



scale-free *exponent*

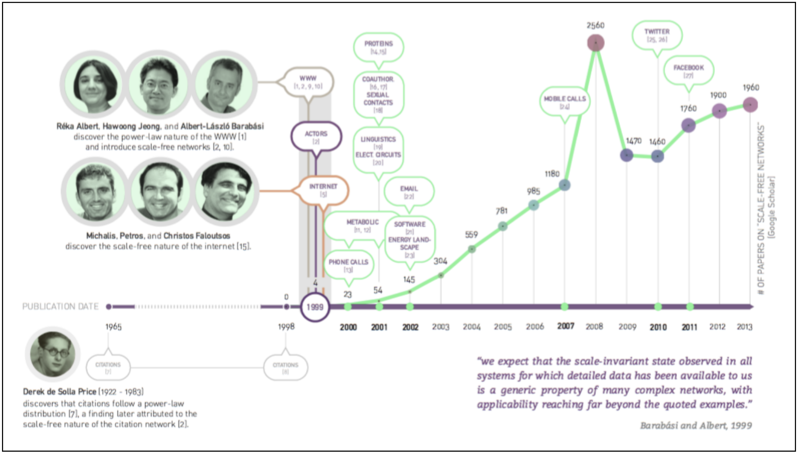


no graphical $\{k\}$ for $\gamma < 2$ $\qquad$ $n = (k_{max} / k_{min})^{\gamma - 1}$ *nonexistent* for $\gamma \gg 3$

scale-free *distributions*

NAME	$p_x/p(x)$	$\langle x \rangle$	$\langle x^2 \rangle$
Poisson (discrete)	$e^{-\mu}\mu^x/x!$	μ	$\mu(1+\mu)$
Exponential (discrete)	$(1-e^{-\lambda})e^{-\lambda x}$	$1/(e^{\lambda}-1)$	$(e^{\lambda}+1)/(e^{\lambda}-1)^2$
Exponential (continuous)	$\lambda e^{-\lambda x}$	$1/\lambda$	$2/\lambda^2$
Power law (discrete)	$x^{-\alpha}/\zeta(\alpha)$	$\begin{cases} \zeta(\alpha-2)/\zeta(\alpha), & \text{if } \alpha > 2 \\ \infty, & \text{if } \alpha \leq 1 \end{cases}$	$\begin{cases} \zeta(\alpha-1)/\zeta(\alpha), & \text{if } \alpha > 1 \\ \infty, & \text{if } \alpha \leq 2 \end{cases}$
Power law (continuous)	$\alpha x^{-\alpha}$	$\begin{cases} \alpha/(\alpha-1), & \text{if } \alpha > 2 \\ \infty, & \text{if } \alpha \leq 1 \end{cases}$	$\begin{cases} \alpha/(\alpha-2), & \text{if } \alpha > 1 \\ \infty, & \text{if } \alpha \leq 2 \end{cases}$
Power law with cutoff (continuous)	$\frac{\lambda^{1-\alpha}}{\Gamma(1-\alpha)} x^{-\alpha} e^{-\lambda x}$	$\lambda^{-1}\frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha)}$	$\lambda^{-2}\frac{\Gamma(3-\alpha)}{\Gamma(1-\alpha)}$
Stretched exponential (continuous)	$\beta\lambda^{\beta}x^{\beta-1}e^{-(\lambda x)^{\beta}}$	$\lambda^{-1}\Gamma(1+\beta^{-1})$	$\lambda^{-2}\Gamma(1+2\beta^{-1})$
Log-normal (continuous)	$\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\ln x-\mu)^2/(2\sigma^2)}$	$e^{\mu+\sigma^2/2}$	$e^{2(\mu+\sigma^2)}$
Normal (continuous)	$\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/(2\sigma^2)}$	μ	$\mu^2+\sigma^2$

scale-free *history*



scale-free *references*

A.-L. Barabási and R. Albert.
Emergence of scaling in random networks.
Science, 286(5439):509–512, 1999.

A.-L. Barabási.
Network Science.
Cambridge University Press, Cambridge, 2016.

Wouter de Nooy, Andrej Mrvar, and Vladimir Batagelj.
Exploratory Social Network Analysis with Pajek: Expanded and Revised Second Edition.
Cambridge University Press, Cambridge, 2011.

David Easley and Jon Kleinberg.
Networks, Crowds, and Markets: Reasoning About a Highly Connected World.
Cambridge University Press, Cambridge, 2010.

Ernesto Estrada and Philip A. Knight.
A First Course in Network Theory.
Oxford University Press, 2015.

P. Erdős and A. Rényi.
On random graphs I.
Publ. Math. Debrecen, 6:290–297, 1959.

Michalis Faloutsos, Petros Faloutsos, and Christos Faloutsos.
On power-law relationships of the Internet topology.
Comput. Commun. Rev., 29(4):251–262, 1999.

Mark E. J. Newman.
Networks.
Oxford University Press, Oxford, 2nd edition, 2018.

scale-free *references*

D. J. de Solla Price.
Networks of scientific papers.
Science, 149:510–515, 1965.