preferential attachment

introduction to *network analysis* (*ina*)

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preferential *attachment*

— *generative models* reason about *network evolution*

— *cumulative advantage* process of *Price model* [Pri76]

— *preferential attachment* of *Barabási-Albert model* [BA99]

Pólya process Yule process Zipf's law Matthew effect
rich-get-richer proportional growth cumulative advantage

see preferential attachment model NetLogo demo

Derek de Solla Price

Albert-László Barabási

Réka Albert

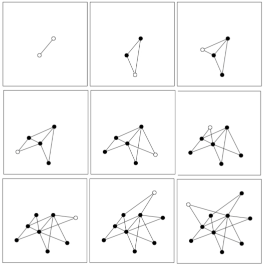
preferential $G(n, c, a)$ *model*

- $G(n, c, a)$ *cumulative advantage* model [Pri76]
- each new node i *forms* $k_i^{out} = c > 0$ *directed links*
- node j *receives link with probability* $\sim k_j^{in} + a = q_j + a > 0$

n, c, a given $\qquad p_q$ *unknown*

input parameters n, c, a
output *directed* graph G

1: $G \leftarrow \geq c$ isolated nodes
2: **while not** G has n nodes **do**
3: add node i to G
4: **for** c **times do**
5: add link (i, j) with $\sim q_j + a$
6: **return** G



preferential $G(n, c, a)$ *equation*

- *master equation* for *in-degree distribution* $p_q(n)$
  - $p_q(n)$ is in-degree distribution p_q *at time* n

$$\frac{q_i+a}{\sum_i q_i+a} = \frac{q_i+a}{n(c+a)} \qquad cnp_q(n)\frac{q+a}{n(c+a)} = \frac{c(q+a)}{c+a}p_q(n)$$

$$(n+1)p_q(n+1) = np_q(n) + \frac{c(q-1+a)}{c+a}p_{q-1}(n) - \frac{c(q+a)}{c+a}p_q(n)$$

$$(n+1)p_0(n+1) = np_0(n) + 1 - \frac{ca}{c+a}p_0(n)$$

- *power-law in-degree distribution* $p_q \sim q^{-\gamma}$ with $\gamma > 2$
  - p_q is in-degree distribution *in limit* $n \to \infty$

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t \qquad B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \simeq x^{-y}\Gamma(y)$$

$$p_q = \frac{q+a-1}{q+a+1+a/c}p_{q-1} = \cdots = \frac{B(q+a, 2+a/c)}{B(a, 1+a/c)} \sim q^{-2-a/c}$$

$$p_0 = \frac{1+a/c}{a+1+a/c}$$

preferential $G(n,c)$ *model*

- $G(n,c)$ *preferential attachment* model [BA99]
- each new node i *forms* $c > 0$ *undirected links*
- node j *receives links with probability* $\sim k_i$

n, c given $\qquad p_k$ *unknown*

input parameters n, c
output *undirected* graph G

```
G ← c connected nodes
while not G has n nodes do
    add node i to G
    for c times do
        add link {i, j} with ~ k_j
return G
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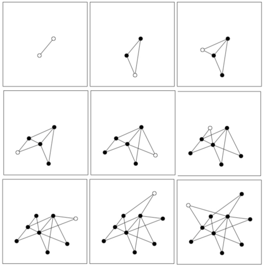


preferential $G(n, c)$ *equation*

- *undirected* $G(n, c)$ is *directed* $G(n, c, c)$ for $k_i = q_i + c$
- *same master equation* for *in-degree distribution* p_q
  - p_q is in-degree distribution *in limit* $n \to \infty$

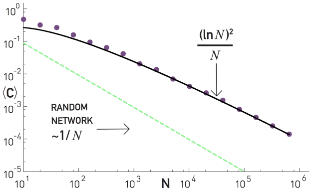
$$p_q = \frac{B(q+c, 2+c/c)}{B(c, 1+c/c)} = \frac{B(q+c, 3)}{B(c, 2)} \sim q^{-3}$$

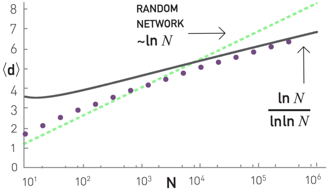
- *power-law degree distribution* $p_k \sim k^{-3}$
  - p_k is degree distribution *in limit* $n \to \infty$

$$p_k = \frac{B(k, 3)}{B(c, 2)} = \cdots = \frac{2c(c+1)}{k(k+1)(k+2)} \sim k^{-3}$$

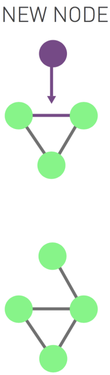
preferential $\neg$*small-world*

- *random graphs* are *"small-world"* as $\langle d\rangle \simeq \frac{\ln n}{\ln\langle k\rangle}$
- *random graphs* are *not small-world* as $\langle C\rangle = \frac{\langle k\rangle}{n-1}$
- *scale-free networks* $\gamma = 3$ are *"small-world"* as $\langle d\rangle \sim \frac{\ln n}{\ln\ln n}$
- $G(n,c)$ *scale-free model* is *not small-world* as $\langle C\rangle \simeq \frac{(\ln n)^2}{n}$

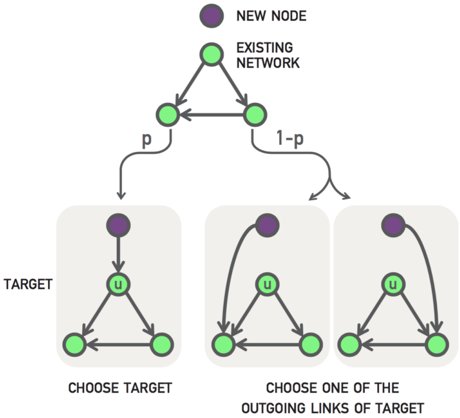




preferential *models*

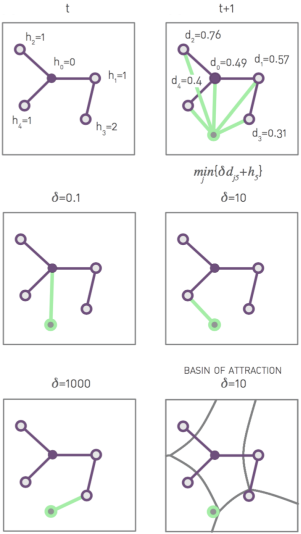


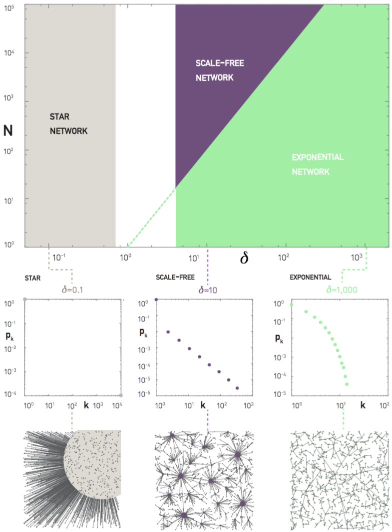
link selection [DM02]



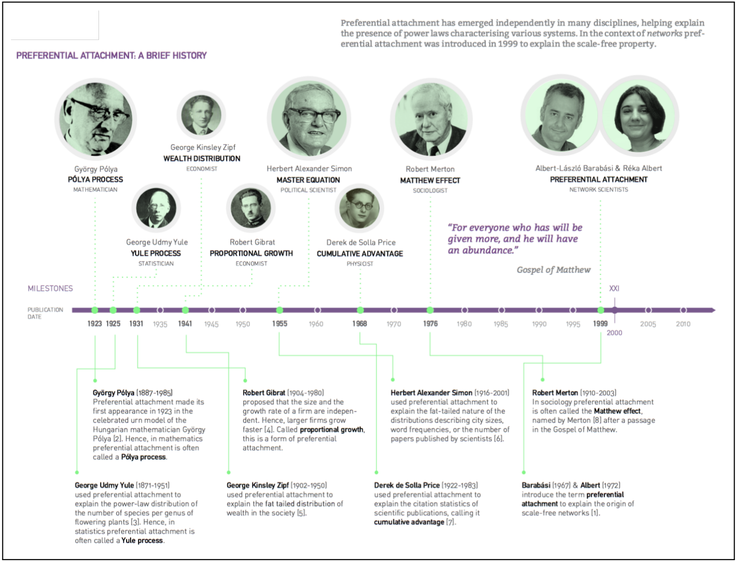
random *link copying* model [KKR+99]

preferential *optimization*





preferential *history*



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