on convexity in complex networks

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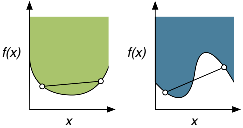
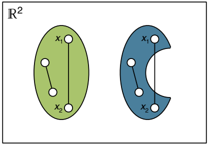
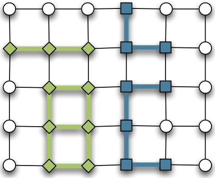
ARS '17

definitions of convexity

convex/non-convex real functions, sets in $\mathbb{R}^2$ & subgraphs

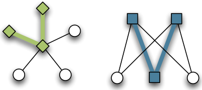


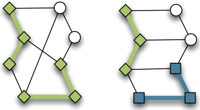




disconnected $\supseteq$ connected $\supseteq$ **induced** $\supseteq$ isometric $\supseteq$ **convex** subgraphs

connected subgraphs induced on simple undirected graph $\rightarrow$





convexity **in** networks?

(**sna**) k-clubs/clans are convex k-cliques

(**cd**) community often defined as "convex" subgraph

— **subset** S is convex if it induces convex **subgraph**

— convex **hull** $\mathcal{H}(S)$ is smallest convex subset including S

hull number $= \min\{|S| : \mathcal{H}(S)$ includes n nodes$\}$ (**Everett & Seidman, 1985**)

↑ hull number measures how **quickly** convex subsets can grow

↓ how **slowly** randomly grown convex subsets expand

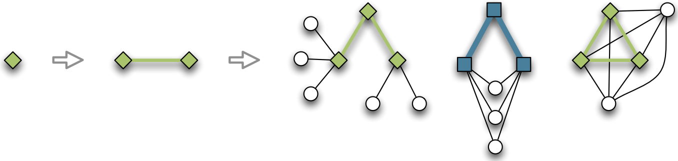
expansion of convex subsets

grow subset S by one node & **expand** S to convex hull $\mathcal{H}(S)$

— $S = \{\text{random node } i\}$

— until S contains n nodes:

1. select $i \notin S$ by random edge
2. expand $S = \mathcal{H}(S \cup \{i\})$

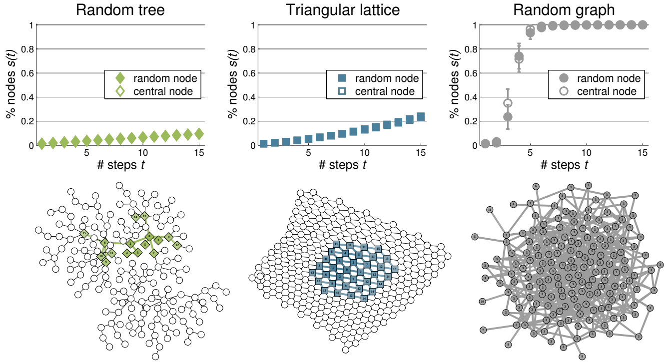


S quantifies (locally) **tree-like**/**clique-like** structure of graphs

convex expansion in graphs

$s(t)$ = fraction of nodes in S after t expansion steps

$s(t) = (t+1)/n$ in **convex** graphs & $s(t) \gg t/n$ in **non-convex** graphs



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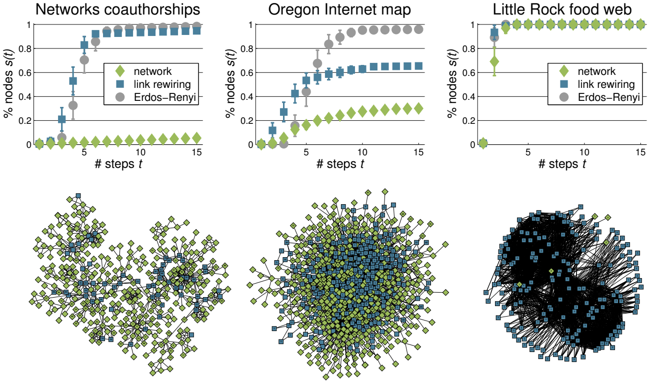

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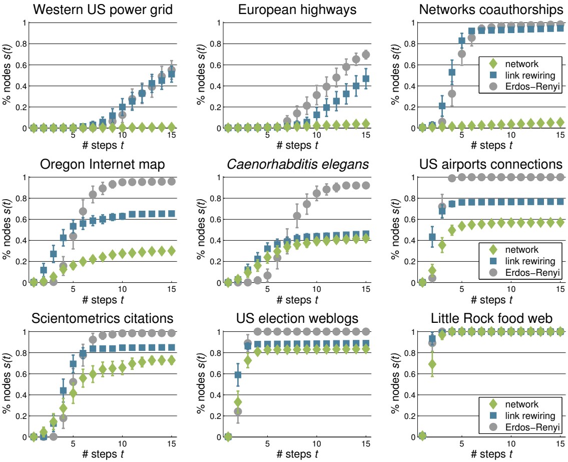
convex expansion in networks

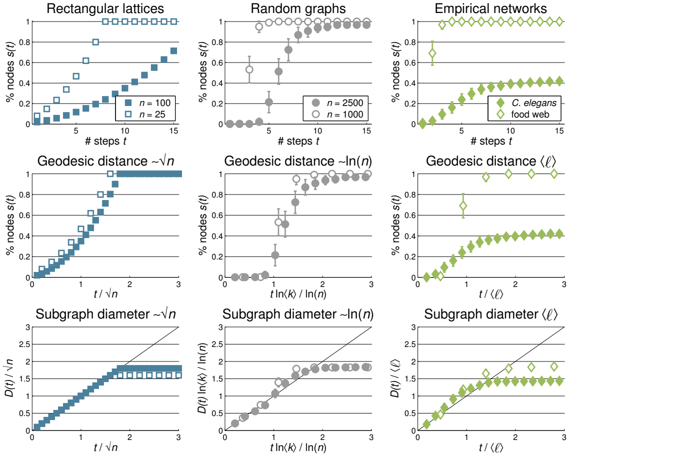
convex infrastructure and collaboration & **non-convex** food web



random **graphs** fail to reproduce convexity in empirical **networks**

when/why sudden expansion?

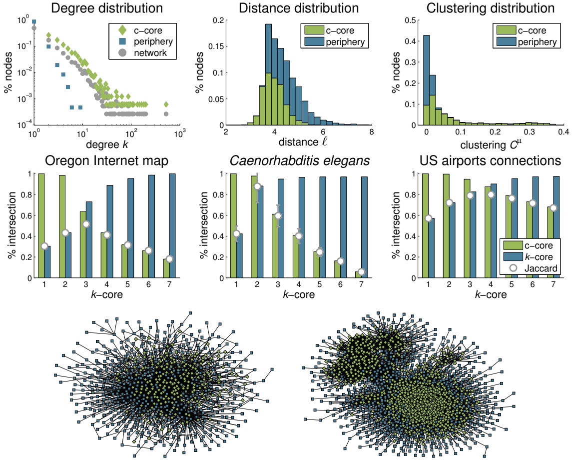
(**why**) steps $t \approx$ diameter $D(t) > $ distance $\langle \ell \rangle$ (**when**)



random graphs **convex** for $< \mathcal{O}(\ln n)$ & **non-convex** for $> \mathcal{O}(\ln^2 n)$

when/why expansion settles?

(**when**) S extends to c-core (**why**) smallest convex subset includ. S



core-periphery networks have **convex** periphery & **non-convex** c-core

global measure c-convexity

$$X_c = 1 - \sum_{t=1}^{n-1} \sqrt[c]{\max(s(t) - s(t-1) - 1/n, 0)} \qquad X_c \geq X_c^{\mathrm{RW}} \geq X_c^{\mathrm{ER}}$$

X_c highlights **tree-like**/**clique-like** networks (cliques connected tree-like)

	X_1	X_1^{RW}	X_1^{ER}	$X_{1.1}$	$X_{1.1}^{\mathrm{RW}}$	$X_{1.1}^{\mathrm{ER}}$
Western US power grid	0.95	0.32	0.24	0.91	0.10	0.01
European highways	0.66	0.23	0.27	0.44	−0.02	0.06
Networks coauthorships	0.91	0.09	0.06	0.83	−0.05	−0.09
Oregon Internet map	0.68	0.36	0.06	0.53	0.20	−0.09
Caenorhabditis elegans	0.57	0.54	0.07	0.43	0.40	−0.13
US airports connections	0.43	0.24	0.00	0.30	0.16	−0.07
Scientometrics citations	0.24	0.16	0.02	0.04	0.00	−0.13
US election weblogs	0.17	0.12	0.00	0.06	0.04	−0.08
Little Rock food web	0.03	0.03	0.02	−0.06	−0.02	−0.02

X_c measures **global** & **regional** (periphery) convexity in networks

local measure of convexity

$$L_c = 1 + \max\{\, t \mid s(t) < (t+c+1)/n \,\} \qquad L_1 \leq L_1^{\mathrm{ER}} \approx \ln n / \ln \langle k \rangle$$

L_c highlights locally **tree-like**/**clique-like** networks & random graphs

	L_t	L_t^{ER}	L_1	L_1^{ER}	$\ln n / \ln \langle k \rangle$
Western US power grid	14	9	6	9	8.66
European highways	16	7	7	7	7.54
Networks coauthorships	17	4	7	4	3.77
Oregon Internet map	3	4	3	4	4.40
Caenorhabditis elegans	2	5	2	5	5.79
US airports connections	2	3	2	3	2.38
Scientometrics citations	3	4	3	4	4.30
US election weblogs	2	2	2	2	2.15
Little Rock food web	2	2	2	2	1.59

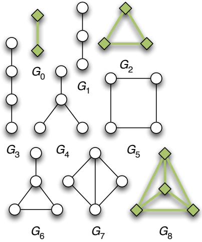
L_c measures **local** & **absolute** (tree/clique) convexity in networks

probability of convex subgraphs

P = probability that random $G_{1\text{-}8}$ convex $\quad P \leq P^{\mathrm{ER}}$

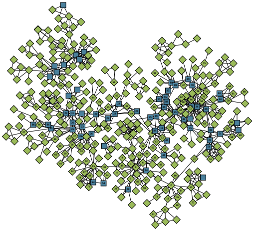
P highlights locally **tree-like**/**clique-like** networks & random graphs

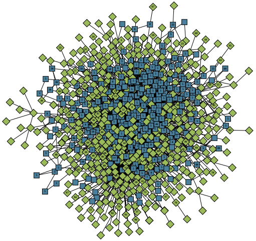
	P	P^{ER}	$\ln n / \ln \langle k \rangle$
Western US power grid	77.0%	99.4%	8.66
European highways	83.2%	97.6%	7.54
Networks coauthorships	53.3%	71.3%	3.77
Oregon Internet map	56.0%	86.4%	4.40
Caenorhabditis elegans	77.8%	97.6%	5.79
US airports connections	5.5%	12.9%	2.38
Scientometrics citations	30.5%	89.2%	4.30
US election weblogs	2.7%	6.0%	2.15
Little Rock food web	2.2%	0.3%	1.59

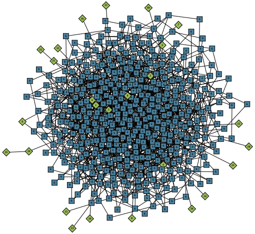


P measures **local** (up to **4 nodes**) convexity in networks

convexity in networks







global convexity

tree/clique-like networks

regional convexity

core-periphery networks etc.

local convexity

random graphs

$< \ln n / \ln \langle k \rangle$

c-core $\neq$ k-cores & **c-convexity** $\neq$ standard measures

robustness, navigation, optimization, sampling, comparison etc.

arXiv:1608.03402v3

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