3 forms of **convexity** in **graphs** & **networks**

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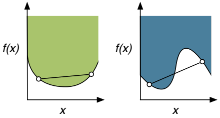
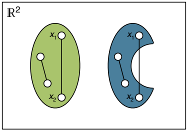
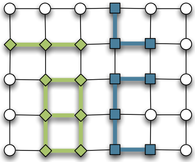
COSTNET '17

definitions of convexity

convex/**non-convex** real functions, sets in $\mathbb{R}^2$ & subgraphs







disconnected ⊇ connected ⊇ **induced** ⊇ isometric ⊇ **convex** subgraphs

(**sna**) k-clubs & k-clans are **convex** k-cliques

(**def**) **subset** S is convex if it induces convex **subgraph**

(**def**) convex **hull** $\mathcal{H}(S)$ is smallest convex subset including S

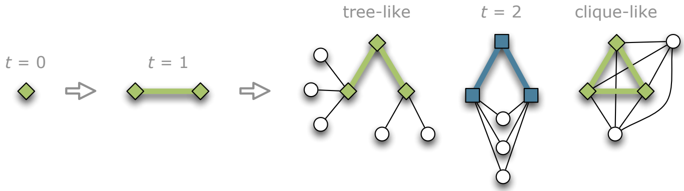
expansion of convex subsets

grow subset S by one node & **expand** S to convex hull $\mathcal{H}(S)$

— $S = \{\text{random node } i\}$

— until S contains n nodes:

1. select $i \notin S$ by random edge
2. expand $S = \mathcal{H}(S \cup \{i\})$

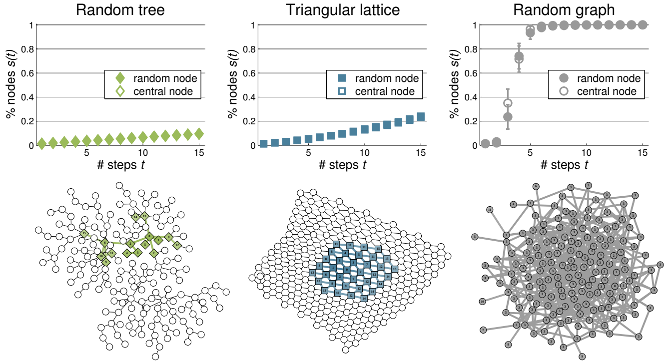


S quantifies (locally) **tree-like**/**clique-like** structure of graphs

convex expansion in graphs

$s(t)$ = average fraction of nodes in S after t expansion steps

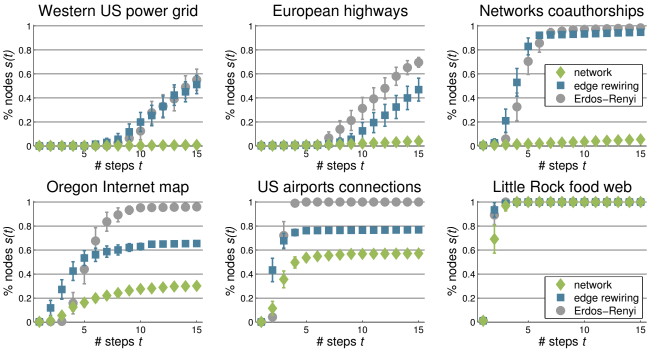
$s(t) = (t+1)/n$ in **convex** & $s(t) \gg (t+1)/n$ in **non-convex** graphs



$s(t)$ quantifies (locally) **tree-like**/**clique-like** structure of graphs

convex expansion in networks

convex infrastructure and collaborations & **non-convex** food web



random **graphs** fail to reproduce convexity in empirical **networks**

random graphs **convex** for $< \mathcal{O}(\ln n)$ & **non-convex** for $> \mathcal{O}(\ln^2 n)$

core-periphery networks have **convex** periphery & **non-convex** c-core

global measure c-convexity

$$X_c = 1 - \sum_{t=1}^{n-1} \sqrt[c]{\max(\Delta s(t) - 1/n, 0)} \qquad X_c \geq X_c^{\mathrm{RW}} \geq X_c^{\mathrm{ER}}$$

X_c highlights **tree-like**/**clique-like** networks (cliques connected tree-like)

	X_1	X_1^{RW}	X_1^{ER}	$X_{1.1}$	$X_{1.1}^{\mathrm{RW}}$	$X_{1.1}^{\mathrm{ER}}$
Western US power grid*	0.95	0.32	0.24	0.91	0.10	0.01
European highways*	0.66	0.23	0.27	0.44	−0.02	0.06
Networks coauthorships	0.91	0.09	0.06	0.83	−0.05	−0.09
Oregon Internet map	0.68	0.36	0.06	0.53	0.20	−0.09
Caenorhabditis elegans	0.57	0.54	0.07	0.43	0.40	−0.13
US airports connections	0.43	0.24	0.00	0.30	0.16	−0.07
Scientometrics citations	0.24	0.16	0.02	0.04	0.00	−0.13
US election weblogs	0.17	0.12	0.00	0.06	0.04	−0.08
Little Rock food web	0.03	0.03	0.02	−0.06	−0.02	−0.02

X_c measures **global** & **regional** (periphery) convexity in networks

local measure of convexity

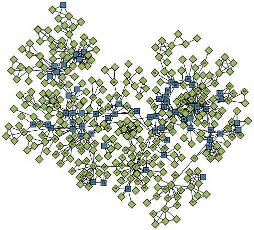
$$L_c = 1 + \max\{\, t \mid s(t) < (t + c + 1)/n \,\} \qquad L_1 \leq L_1^{\mathrm{ER}} \approx \ln n / \ln \langle k \rangle$$

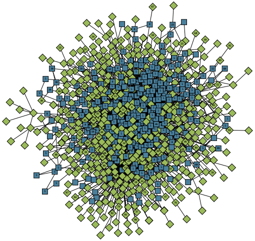
L_c highlights locally **tree-like**/**clique-like** networks & random graphs

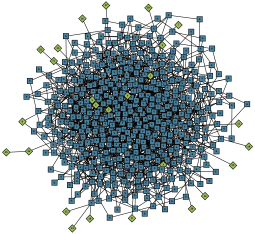
	L_t	L_t^{ER}	L_1	L_1^{ER}	$\ln n / \ln \langle k \rangle$
Western US power grid	14	9	6	9	8.66
European highways	16	7	7	7	7.54
Networks coauthorships	17	4	7	4	3.77
Oregon Internet map	3	4	3	4	4.40
Caenorhabditis elegans	2	5	2	5	5.79
US airports connections	2	3	2	3	2.38
Scientometrics citations	3	4	3	4	4.30
US election weblogs	2	2	2	2	2.15
Little Rock food web	2	2	2	2	1.59

L_c measures **local** & **absolute** (tree/clique) convexity in networks

convexity in graphs & networks







global convexity

tree/clique-like networks

regional convexity

core-periphery networks etc.

local convexity

random graphs $< \ln n / \ln \langle k \rangle$

***c*-convexity** $\neq$ standard measures & **c-core** $\neq$ k-cores

robustness, navigation, optimization, abstraction, comparison etc.

to be continued...

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Marc & Šubelj (2017) Convexity in complex networks, *Network Science*, pp. 27

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convex skeletons of networks

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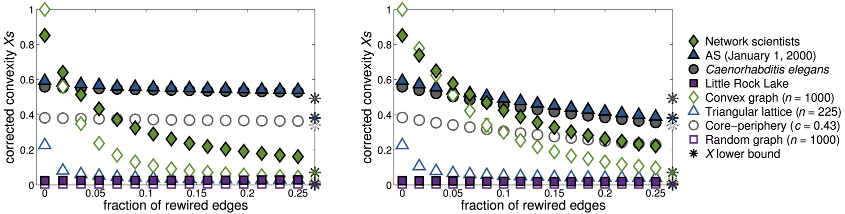
convexity under randomization

$$Xs = s - \sum_{t=1}^{sn-1} \sqrt[c]{\max(s\Delta s(t) - 1/n, 0)} \qquad s = \text{fraction of nodes in LCC}$$

Xs under **degree-preserving/full randomization** by edge rewiring

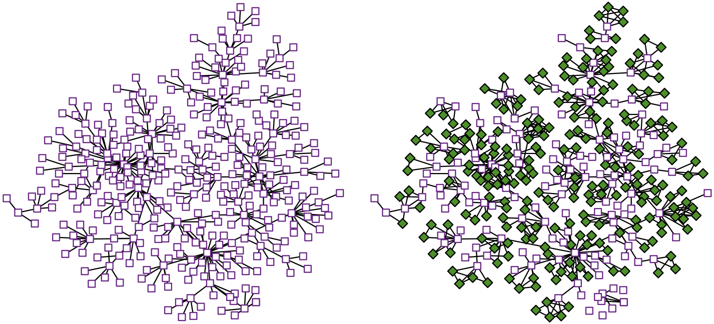


Xs very **sensitive** to **random perturbations** of network structure

convex skeletons of networks

convex skeleton = largest high-*Xs* subnetwork (every *S* is convex)

spanning tree & **convex skeleton** of network scientists coauthorships



convex skeleton is **tree** of **cliques** extracted by targeted edge removal

statistics of convex skeletons

$$\langle C \rangle = \frac{1}{n} \sum_i \frac{2t_i}{k_i(k_i-1)} \qquad \langle \sigma \rangle = \frac{2}{n(n-1)} \sum_{i<j} \sigma_{ij} \qquad Xs = \ldots$$

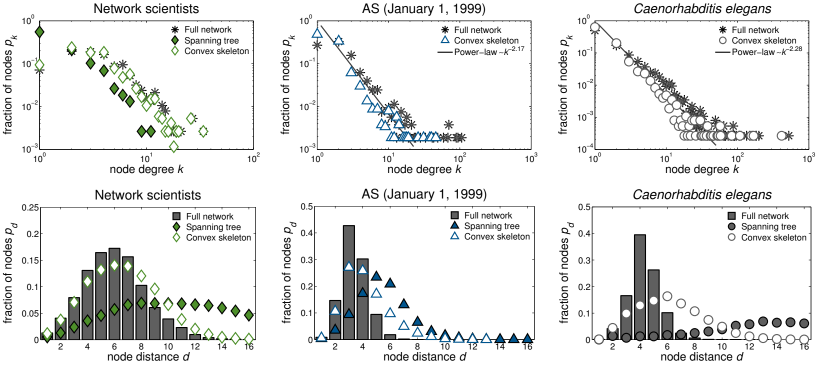
statistics of **convex skeletons** & **spanning trees** of networks

	clustering $\langle C \rangle$			geodesics $\langle \sigma \rangle$			convexity Xs		
	N	CS	ST	N	CS	ST	N	CS	ST
Jazz musicians	0.62	0.81	0.00	9.71	1.97	1.00	0.12	0.84	1.00
Network scientists	0.74	0.75	0.00	2.66	1.47	1.00	0.85	0.95	1.00
Computer scientists	0.48	0.54	0.00	4.08	1.42	1.00	0.64	0.95	1.00
Plasmodium falciparum	0.02	0.07	0.00	3.71	1.77	1.00	0.43	0.95	1.00
Saccharomyces cerevisiae	0.07	0.10	0.00	2.58	1.19	1.00	0.68	0.88	1.00
Caenorhabditis elegans	0.06	0.12	0.00	6.79	3.03	1.00	0.56	0.85	1.00
AS (January 1, 1998)	0.18	0.21	0.00	3.87	2.32	1.00	0.66	0.91	1.00
AS (January 1, 1999)	0.18	0.27	0.00	3.54	2.05	1.00	0.49	0.95	1.00
AS (January 1, 2000)	0.20	0.25	0.00	4.81	3.07	1.00	0.59	0.90	1.00
Little Rock Lake	0.32	0.69	0.00	22.13	4.32	1.00	0.02	0.82	1.00
Florida Bay (wet)	0.33	0.79	0.00	9.17	1.37	1.00	0.03	0.92	1.00
Florida Bay (dry)	0.33	0.82	0.00	9.37	1.65	1.00	0.03	0.93	1.00

convex skeleton is generalization of **spanning tree** retaining **clustering**

distributions of convex skeletons

distributions of **convex skeletons** & **spanning trees** of networks

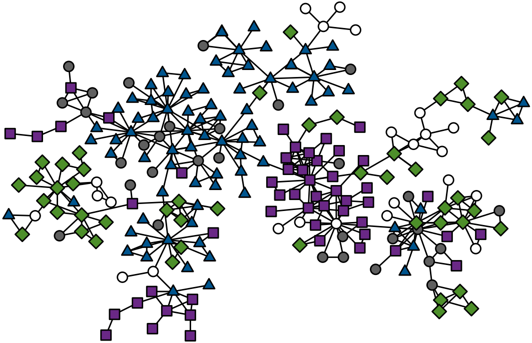


convex skeletons retain distributions in contrast to **spanning trees**

convex skeletons of coauthorships

convex skeleton ∼ network abstraction technique

convex skeleton of Slovenian **computer scientists** coauthorships



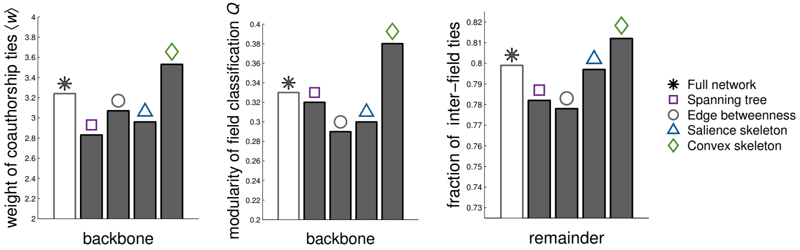
computer theory (◆), information systems (■), intelligent systems (▲),

programming technologies (○) & other (●)

network backbones of coauthorships

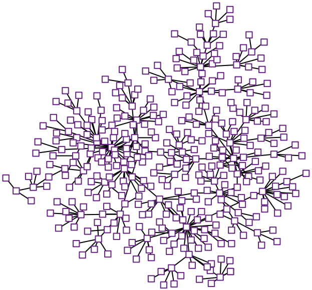
convex skeleton $\gg$ high-betweenness & high-salience backbones

properties of **backbones** of Slovenian **computer scientists** coauthorships



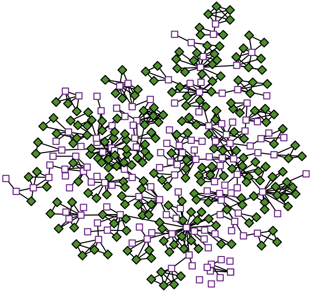
convex skeletons enhance properties in contrast to **other backbones**

convex skeletons of networks



spanning **tree**

tree w/o cliques



convex skeleton

tree w/ cliques

convex skeleton $\gg$ backbones & **c-centrality** $\neq$ centralities

abstraction, sampling, visualization, modeling, dynamics etc.

thank you!

arXiv:**1709.00255v2**

Šubelj (2017) Convex skeletons of complex networks, pp. 19

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