network abstraction with backbones and skeletons:

spanning trees vs convex skeletons

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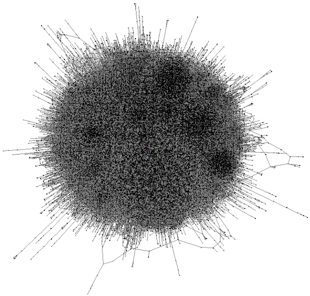
network abstraction

many real **networks** are too **large**/**dense**/**complex**/**noisy**

for superlinear algorithms, GPU memory, clear visualizations, etc.

abstraction techniques try to simply network

preserving network structure/dynamics as much as possible



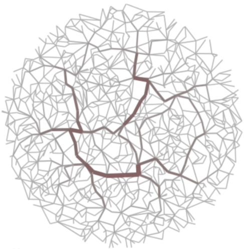
backbones and skeletons

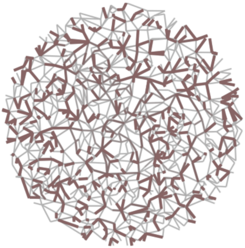
network backbone keeps most important edges (e.g., information flow)

sparsification technique that <u>removes</u> as many edges as possible

network skeleton preserves overall structure (e.g., with simpler graph)

simplification technique that <u>retains</u> as many edges as possible





† Grady et al. (2012) *Nature Communications* **3**, 864.

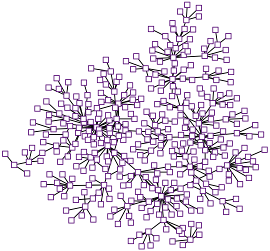
network skeletons

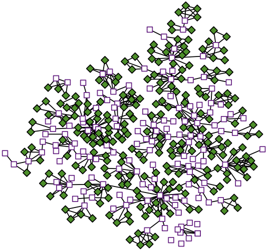
spanning tree is smallest connected graph on all nodes

network simplification technique that retains tree (of edges)

convex skeleton is tree connecting network cliques

network simplification technique that retains tree of cliques





network structure

simple undirected **unweighted network** (n nodes and m edges)

$$\langle k\rangle = \frac{1}{n}\sum_i k_i = \frac{2m}{n}$$

most large real networks are almost **connected and sparse**

$$\rho = \frac{m}{\binom{n}{2}} = \frac{\langle k\rangle}{n-1} \to 0 \text{ when } n \to \infty$$

small-world networks have high clustering and short distances

$$\langle C\rangle \gg \frac{\langle k\rangle}{n-1} \text{ and } \langle d\rangle \sim \log n$$

scale-free networks have heavy-tailed degree distribution

$$p_k \sim k^{-\gamma} \text{ for } \gamma > 1$$

spanning trees

network abstraction with **spanning trees**

$$\langle k \rangle = \frac{1}{n} \sum_i k_i = \frac{2m}{n}$$

spanning trees preserve **connectivity and sparsity**

$$m = n - 1 \text{ and } \langle k \rangle = 2 - \frac{2}{n}$$

spanning trees **lack clustering** or longer cycles

$$\langle C \rangle = 0 \text{ by definition}$$

are spanning trees also **small-world and scale-free?**

$$\langle d \rangle \sim \log n \text{ and } p_k \sim k^{-\gamma}?$$

† $\langle d \rangle \sim \sqrt{n}$ in random trees (Reńyi and Szekeres, 1967)

spanning algorithms

Kruskal's algorithm

1. start with forest of trees each consisting of one node
2. merge trees until only one tree remains

Prim's algorithm

1. start with single tree consisting of seed node
2. add one new node at each step

breadth-first search

1. start with single tree consisting of seed node
2. add new neighbors at each step (in breadth-first order)

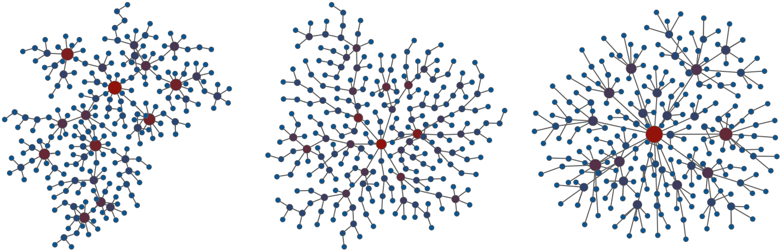
other algorithms

depth-first or beam search, Sollin's algorithm, etc.

wiring diagrams

examples of spanning trees of random graph

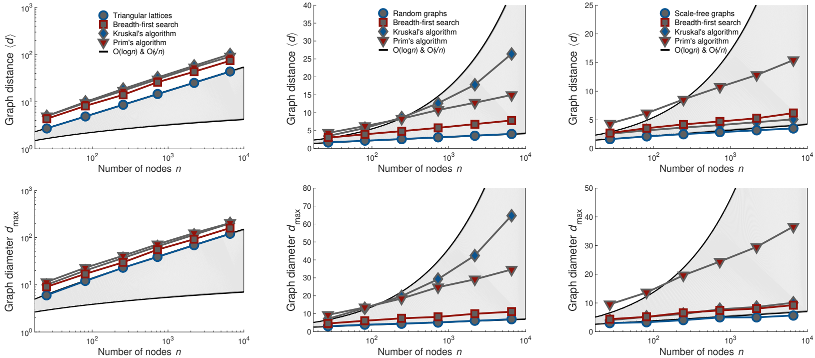
Kruskal's algorithm, Prim's algorithm and breadth-first search



synthetic graphs

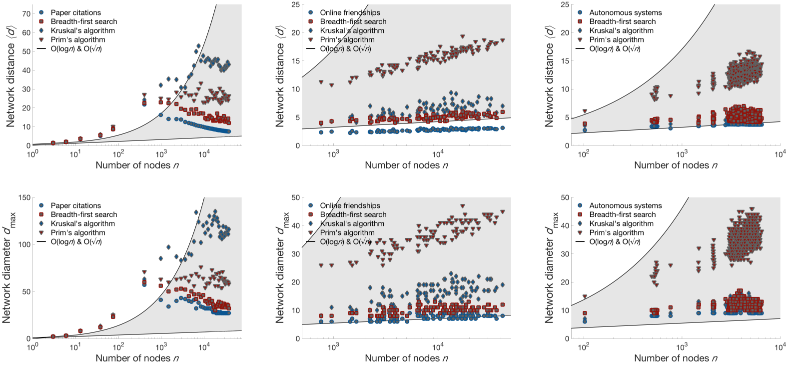
breadth-first search **preserves distances** $\langle d \rangle$ in synthetic graphs

$\langle d \rangle \sim \sqrt{n}$ in lattices, $\langle d \rangle \sim \log n$ in random and $\langle d \rangle \sim \frac{\log n}{\log \log n}$ in scale-free graphs



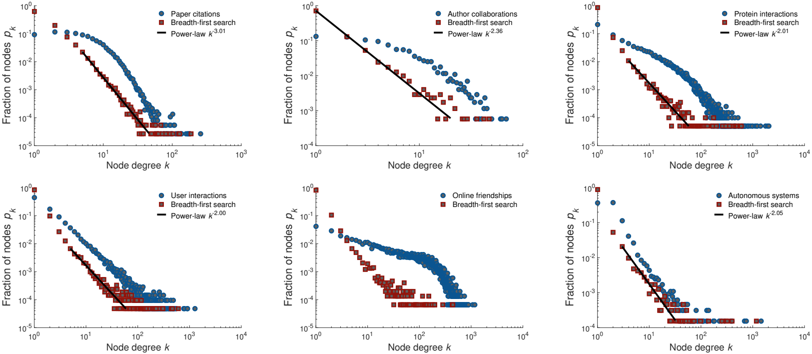
breadth-first search **preserves short distances** $\langle d \rangle$ in real networks

$\langle d \rangle \sim \log n$ in small-world and $\langle d \rangle \sim \log\log n$ in ultra small-world networks



scale-free networks

breadth-first search **power-law $p_k \sim k^{-\gamma}$** as in real networks

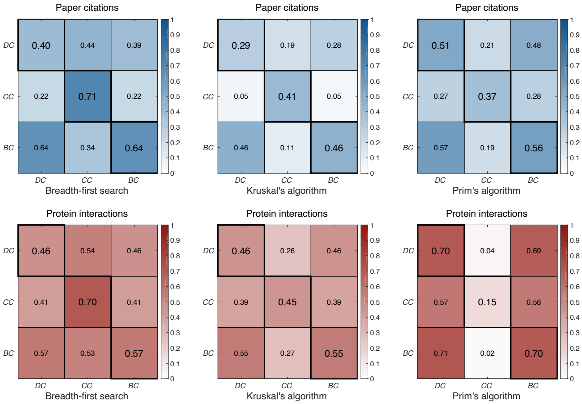


† Clauset et al. (2009) *SIAM Review* **51**(4), 661-703.

node position

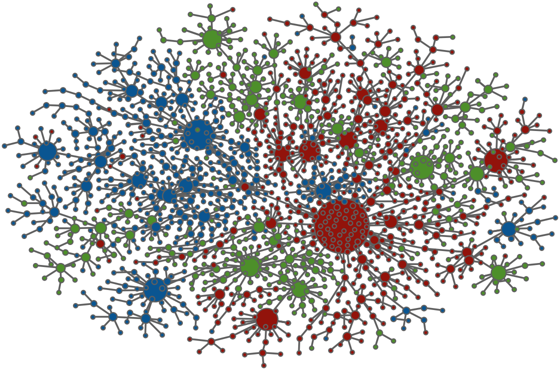
breadth-first search **preserves closeness centrality** of network nodes

correlations btw degree (DC), **closeness** (CC) and betweenness (BC) centrality



breadth-first search tree of Slovenian scientists coauthorships

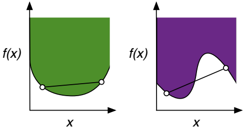
primary disciplines = natural sciences, engineering, medical sciences, etc.

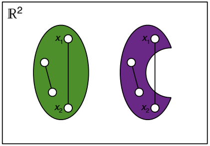


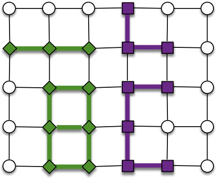
1. backbones/skeletons
2. spanning trees
3. convex skeletons
4. conclusions

convexity in networks

convex/non-convex real functions, sets in $\mathbb{R}^2$ & subgraphs







convex graph $\equiv$ every connected induced subgraph is convex

convexity estimated by simulating growth of convex subgraphs

convex network $=$ tree of cliques (cliques connected tree-like)

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† Marc & Šubelj (2018) *Network Science* **6**(2), 176-203.

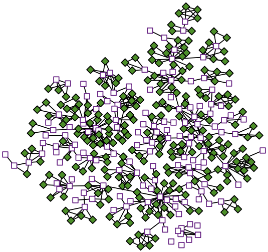
convex skeleton

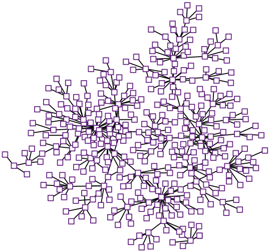
convex skeleton $=$ largest high-convexity subgraph of network

convex skeletons extracted by non-convex edge removal

convex skeleton $\sim$ generalized spanning tree

tree of cliques $\supseteq$ tree (of edges)





convex skeletons

network abstraction with **convex skeletons**

$$\langle k \rangle = \frac{1}{n} \sum_i k_i = \frac{2m}{n}$$

convex skeletons preserve **connectivity and sparsity**

$$m \geq n - 1 \text{ and } \langle k \rangle \geq 2 - \frac{2}{n}$$

convex skeletons preserve **clustering and cliques**

$$\langle C \rangle \gg 0 \text{ by construction}$$

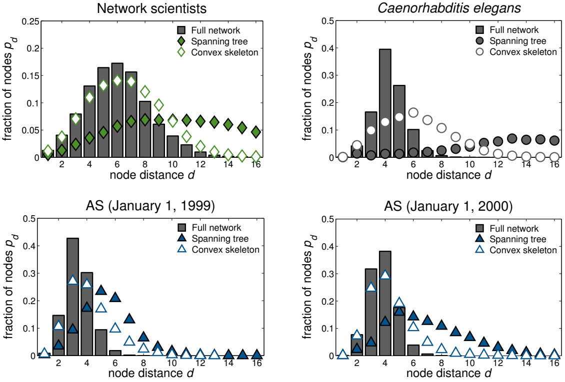
are convex skeletons also **small-world and scale-free?**

$$\langle d \rangle \sim \log n \text{ and } p_k \sim k^{-\gamma}?$$

† $\langle d \rangle \sim \sqrt{n}$ in random trees (Reńyi and Szekeres, 1967)

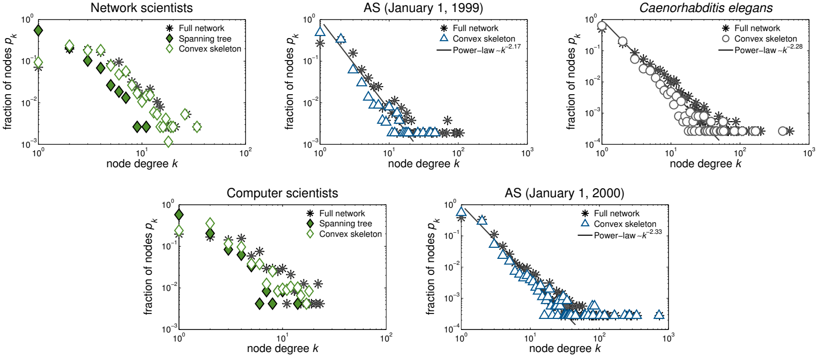
small-world networks

convex skeleton **preserves distance distribution p_d** of real networks



scale-free networks

convex skeleton **preserves degree distribution p_k** of real networks

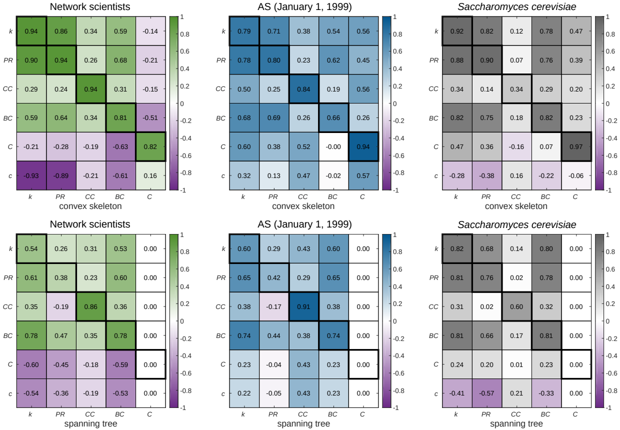


† Clauset et al. (2009) *SIAM Review* **51**(4), 661-703.

node position

convex skeleton **preserves position/centrality** of network nodes

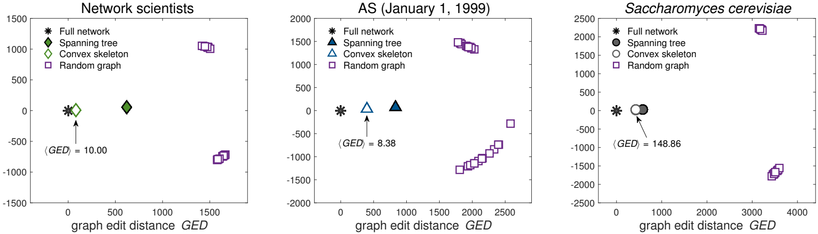
correlations btw degree (k), closeness (CC), betweenness (BC), **clustering (C)**, etc.



edit distances

convex skeleton **provides good model** of real networks

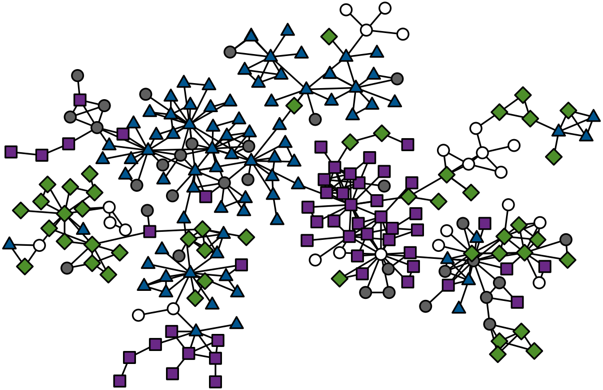
MDS maps of **edit distances** between networks and skeletons



network visualization

convex skeleton of Slovenian computer scientists coauthorships

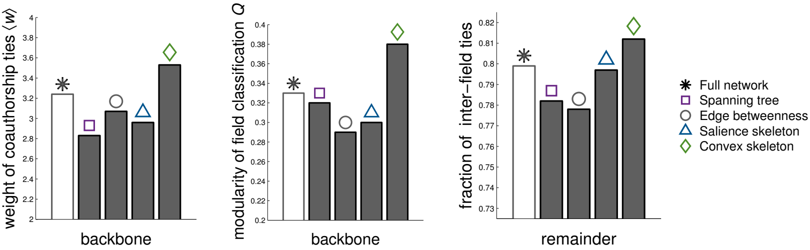
research areas = computer theory, intelligent systems, information technology, etc.



network skeletons

skeletons of Slovenian computer scientists coauthorships

convex skeleton strengthens **desirable properties** of scientists coauthorships



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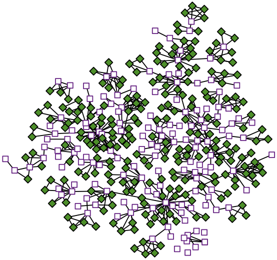
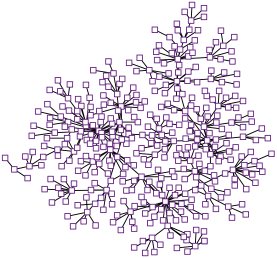
conclusions



convex skeleton

tree with cliques



spanning tree

tree without cliques

quality $\rightarrow$ convex skeleton $\gg$ spanning tree

complexity $\rightarrow$ spanning tree $\ll$ convex skeleton

thank you!

Marc & Šubelj (2018) **Convexity in complex networks**. *Network Science* **6**(2), 176-203.

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